Solutions to the Wave Equation

(Part A – Scalar, One-Dimensional Analysis)

Diffraction and Interferometry

Simple Transverse Waves

One-dimensional wave equation:

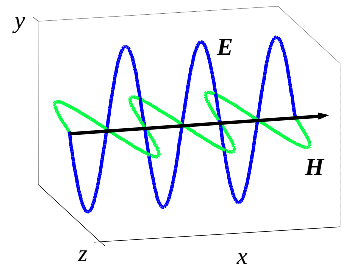
$$\frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} = 0$$

Solution for the real wave traveling in $+x$ direction:

$$E_y(x,t)\,\hat{\boldsymbol{y}} = A_0 \cos(kx - \omega t + \phi)\,\hat{\boldsymbol{y}}$$

if $k = \dfrac{\omega}{c}$.

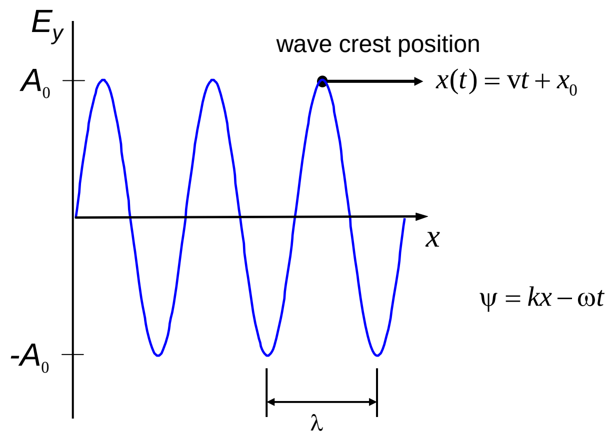
The complete solution involves consideration of the magnetic field. If we take a snapshot in time of the EM wave it may look like:



E ⊥ ***H***

S = ***E*** x ***H***

Simple Transverse Waves



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Complex Notation

- To simplify calculations, complex notation is often used.
- To find the value of the real electric field, simply take the real part of the result.

$$
\begin{aligned}
E_y(x,t)\,\hat{\boldsymbol{y}} &= A_0 \cos(kx - \omega t + \phi)\,\hat{\boldsymbol{y}} \\
&= \mathrm{Re}\left\{A_0 \exp\left[\,j(kx - \omega t + \phi)\right]\right\}\hat{\boldsymbol{y}} \\
&= \mathrm{Re}\left[A_0 e^{j\phi} e^{j(kx-\omega t)}\right]\hat{\boldsymbol{y}} \\
&= \mathrm{Re}\left[U_y(x,t)\right]\hat{\boldsymbol{y}}
\end{aligned}
$$

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Waves Having Same Frequency, but Different Amplitude and Phase

$$U(x,t) = A_0 e^{j\phi_0} e^{j(kx-\omega t)} = A_1 e^{j\phi_1} e^{j(kx-\omega t)} + A_2 e^{j\phi_2} e^{j(kx-\omega t)}$$ (By linear addition.)

$$A_0 e^{j\phi_0} = A_1 e^{j\phi_1} + A_2 e^{j\phi_2}$$
$$= A_1 \cos\phi_1 + A_2 \cos\phi_2 + j\left(A_1 \sin\phi_1 + A_2 \sin\phi_2\right)$$

$A_0, A_1, A_2 \in \mathrm{Re}\{\ \}$

$$\tan\phi_0 = \frac{A_0 \sin\phi_0}{A_0 \cos\phi_0} = \frac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi_1 + A_2 \cos\phi_2}$$

$$I_0 \propto A_0^2 = \left|A_1 e^{j\phi_1} + A_2 e^{j\phi_2}\right|^2 = \boxed{A_1^2 + A_2^2 + 2A_1A_2 \cos\{\phi_2 - \phi_1\}}$$

This is an important result that is the basis for much of the class.

If $A_1 = A_2$,

$$I_0 \propto A_0^2 = 2A_1^2 \left\{1 + \cos\left(\phi_2 - \phi_1\right)\right\} = 4A_1^2 \cos^2\left\{\frac{1}{2}\left(\phi_2 - \phi_1\right)\right\}$$

Combination of Several Waves Having Same Frequency, but Different Amplitude and Phase

$$U(x,t) = A_0 e^{j\phi_0} e^{j(kx-\omega t)} = \sum_{n=1}^{N} A_n e^{j\phi_n} e^{j(kx-\omega t)}$$

$$A_0 e^{j\phi_0} = \sum_{n=1}^{N} A_n e^{j\phi_n}$$

$$\tan\phi_0 = \frac{\sum_{n=1}^{N} A_n \sin\phi_n}{\sum_{n=1}^{N} A_n \cos\phi_n}$$

$$A_0^2 = \boxed{\sum_{n=1}^{N} A_n^2} + \boxed{\sum_{n=1}^{N} \sum_{m=1, m<n}^{N} 2A_nA_m \cos\left(\phi_n - \phi_m\right)}$$

interference term

Term proportional to sum of energy from individual components

Combination of Several Waves Having Same Frequency, but Different Amplitude and Phase

For ϕ_n's random over the observation time,

$$\langle A_0^2 \rangle_t = \left\langle \sum_{n=1}^{N} A_n^2 \right\rangle_t + \left\langle \sum_{n=1}^{N} \sum_{\substack{m=1 \\ m<n}}^{N} 2 A_n A_m \cos(\phi_n - \phi_m) \right\rangle_t$$

$$= \sum_{n=1}^{N} A_n^2 + \sum_{n=1}^{N} \sum_{\substack{m=1 \\ m<n}}^{N} 2 A_n A_m \langle \cos(\phi_n - \phi_m) \rangle_t$$

$$= \sum_{n=1}^{N} A_n^2 = N A^2 \quad (\text{if } A_n = A)$$

$A_0^2 = NA^2$ if $A_n = A$. This is generally referred to as *incoherent* addition.

If $\phi_n = \phi$ = constant over the observation time,

$A_0^2 = \left(\sum_{n=1}^{N} A_n \right)^2$, which is called *coherent* addition.

If all $A_n = A$, $A_0^2 = N^2 A^2$ for coherent addition.

Beats

Consider two waves with the same amplitude, but different frequency.

$$\mathrm{Re}\left[U_a(x,t)\right] = A\cos(k_a x - \omega_a t + \phi_a)$$

$$\mathrm{Re}\left[U_b(x,t)\right] = A\cos(k_b x - \omega_b t + \phi_b)$$

Adding the two waves produces

$$\mathrm{Re}\left[U(x,t)\right] = A\left\{\cos(k_a x - \omega_a t + \phi_a) + \cos(k_b x - \omega_b t + \phi_b)\right\} \quad .$$

Use of the identity $\cos X + \cos Y = 2\cos\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right)$ yields

$$\mathrm{Re}\left[U(x,t)\right] = 2A\left[\cos\left\{\left(\frac{k_a + k_b}{2}\right)x - \left(\frac{\omega_a + \omega_b}{2}\right)t + \left(\frac{\phi_a + \phi_b}{2}\right)\right\} \cdots \right.$$

$$\left. \cos\left\{\left(\frac{k_a - k_b}{2}\right)x - \left(\frac{\omega_a - \omega_b}{2}\right)t + \left(\frac{\phi_a - \phi_b}{2}\right)\right\}\right] \quad .$$

Beats

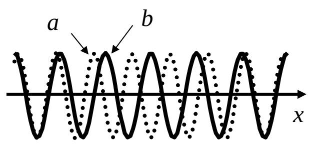
Let

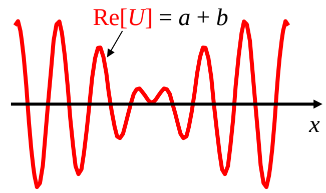
$\omega_\Sigma = \omega_a + \omega_b \quad k_\Sigma = k_a + k_b$ high temporal frequency, short wavelength

$\omega_\Delta = \omega_a - \omega_b \quad k_\Delta = k_a - k_b$ lower temporal frequency, longer wavelength (modulation)

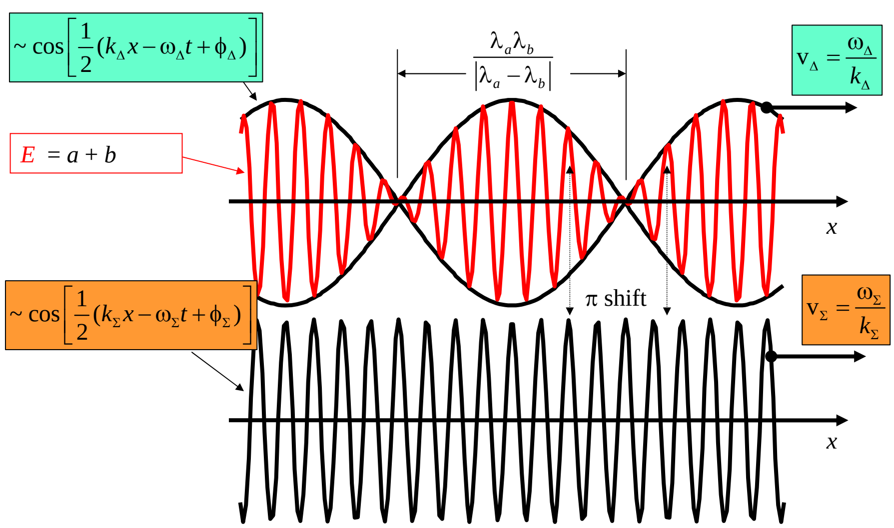
$\phi_\Sigma = \phi_a + \phi_b \quad \phi_\Delta = \phi_a - \phi_b$

$$\mathrm{Re}\left[U(x,t)\right] = E(x,t) = 2A\cos\left[\frac{1}{2}(k_\Sigma x - \omega_\Sigma t + \phi_\Sigma)\right]\cos\left[\frac{1}{2}(k_\Delta x - \omega_\Delta t + \phi_\Delta)\right]$$





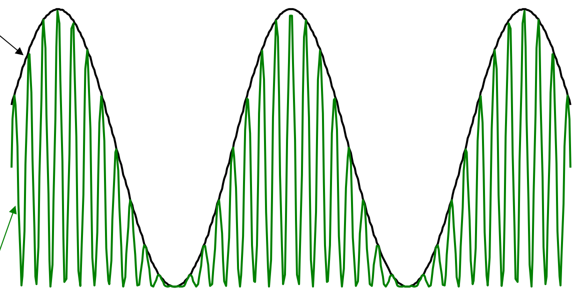
Beats



Beats – Electric Field Squared

$\sim [1+\cos(k_\Delta x - \omega_\Delta t + \phi_\Delta)]$

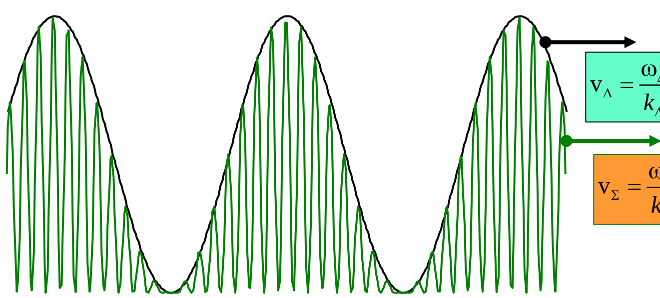
modulation envelope



$$\lambda_{eq} = \frac{\lambda_a \lambda_b}{|\lambda_a - \lambda_b|}$$

$$|U(x,t)|^2 = A^2 [1+\cos(k_\Sigma x - \omega_\Sigma t + \phi_\Sigma)][1+\cos(k_\Delta x - \omega_\Delta t + \phi_\Delta)]$$

Beats – Nondispersive Medium



$$v_\Delta = \frac{\omega_\Delta}{k_\Delta} = \frac{c}{n}$$

$$v_\Sigma = \frac{\omega_\Sigma}{k_\Sigma} = \frac{c}{n}$$

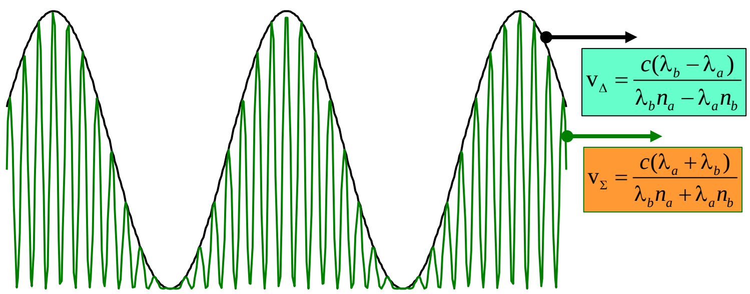
In a nondispersive medium, like air, $\frac{\omega_a}{k_a} = \frac{\omega_b}{k_b} = v = \frac{c}{n}$.

$k_\Sigma = k_a + k_b$ $\qquad$ $k_\Delta = k_a - k_b$

$\omega_\Sigma = \omega_a + \omega_b = \frac{c}{n}(k_a + k_b)$ $\qquad$ $\omega_\Delta = \omega_a - \omega_b = \frac{c}{n}(k_a - k_b)$

Group v_Δ and phase v_Σ velocities are equal for waves traveling in the same direction.

Beats – Dispersive Medium



In a dispersive medium, like glass, $\dfrac{\omega_a}{k_a} = v_a = \dfrac{c}{n_a}$ and $\dfrac{\omega_b}{k_b} = v_b = \dfrac{c}{n_b}$.

$$k_\Sigma = k_a + k_b \qquad k_\Delta = k_a - k_b$$

$$\omega_\Sigma = \omega_a + \omega_b = c\left(\frac{k_a}{n_a} + \frac{k_b}{n_b}\right) \qquad \omega_\Delta = \omega_a - \omega_b = c\left(\frac{k_a}{n_a} - \frac{k_b}{n_b}\right)$$

Group v_Δ and phase v_Σ velocities are not equal for waves traveling in the same direction.

Standing Waves

(Waves traveling in opposite directions)

$$E_a(x,t) = A\cos(kx - \omega t + \phi_a)$$
$$E_b(x,t) = A\cos(kx + \omega t + \phi_b)$$
$$k_a = k_b = k$$
$$\omega_a = -\omega_b = \omega$$
$$E(x,t) = 2A\cos\left[\frac{1}{2}(k_\Sigma x - \omega_\Sigma t + \phi_\Sigma)\right]\cos\left[\frac{1}{2}(k_\Delta x - \omega_\Delta t + \phi_\Delta)\right]$$
$$= 2A\cos\left(kx + \frac{\phi_\Sigma}{2}\right)\cos\left(-\omega t + \frac{\phi_\Delta}{2}\right)$$
$$E^2(x,t) = A^2\left[1 + \cos(2kx + \phi_\Sigma)\right]\left[1 + \cos(-2\omega t + \phi_\Delta)\right]$$
$$I(x) = cn\varepsilon_0\left\langle E^2(x,t)\right\rangle = cn\varepsilon_0 A^2\left[1 + \cos(2kx + \phi_\Sigma)\right]$$

Standing Waves