Summary of OPTI 505R
Chapter 9~14

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✓

good

I. Direct Phase Measurement Interferometry

1.1 Advantages

- High measurement accuracy (>1/1000 fringe, fringe following only 1/10 fringe)
- Rapid measurement
- Good results with low contrast fringes
- Results independent of intensity variations across pupil
- Phase obtained at fixed grid of points
- Easy to use with large solid-state detector arrays

1.2 Four Step Method

$$I(x,y) = I_{dc} + I_{ac}\cos[\underset{\text{measured object phase}}{\phi(x,y)} + \overset{\text{phase shift}}{\phi(t)}]$$

$I_1(x,y) = I_{dc} + I_{ac}\cos[\phi(x,y)]$	$\phi(t) = 0$ (0°)
$I_2(x,y) = I_{dc} - I_{ac}\sin[\phi(x,y)]$	$= \pi/2$ (90°)
$I_3(x,y) = I_{dc} - I_{ac}\cos[\phi(x,y)]$	$= \pi$ (180°)
$I_4(x,y) = I_{dc} + I_{ac}\sin[\phi(x,y)]$	$= 3\pi/2$ (270°)

$$\boxed{\mathrm{Tan}[\phi(x,y)] = \frac{I_4(x,y) - I_2(x,y)}{I_1(x,y) - I_3(x,y)}}$$

Then, the height can be calculated as

$$\mathrm{Height}(x,y) = \frac{\lambda}{4\pi}\phi(x,y)$$

1.3 Phase Shifting Methods

Diffraction grating, Bragg Cell, Moving mirror, Rotating half wave plate, Rotating Polarizer Phase Shifter

1.4 Phase Ambiguities

If we know sign of the Sin and sign of the Cosine the Arc Tangent is calculated modulo 2π.

Removing Phase Ambiguities by

When phase jumps by > π

Add or subtract N2π

Adjust so < π

1.5 Way to fix the vibration problem

- Reduce vibration
- Take data fast
- Measure vibration and introduce vibration 180 degrees out of phase to cancel vibration
- Take all frames at once

1.6 Vertical Scanning Interferometry

Eliminates ambiguities in heights present with monochromatic interferometry

- Best fringe contrast corresponds to zero optical path difference
- Best focus corresponds to zero optical path difference

II. Basic Diffraction Theories

2.1 Huygens Wavelet

Huygens wavelet can be drive using the Dirichlet B.C.m which is shown in the table.

Boundary Conditions	$G(P_0;P_1)$	Across Σ	In Shadow
Kirchhoff[1] (Cauchy)	$-\frac{e^{jkr_{01}}}{4\pi r_{01}}$	U and $\partial U/dn$ same as without screen	$U=0$ $\frac{\partial U}{\partial n}=0$
Neumann	$-\frac{e^{jkr_{01}}}{4\pi r_{01}} - \frac{e^{jkr'_{01}}}{4\pi r'_{01}}$	$\partial U/dn$ same as without screen	$\frac{\partial U}{\partial n}=0$
Dirichlet	$-\frac{e^{jkr_{01}}}{4\pi r_{01}} + \frac{e^{jkr'_{01}}}{4\pi r'_{01}}$	U same as without screen	$U=0$

The resulting Huygens wavelet is

$$h(P_0;P_1) = \frac{\partial}{\partial z_1}\frac{e^{jkr_{01}}}{2\pi r_{01}}$$

$$= \frac{\sqrt{1+(kr_{01})^2}}{2\pi r_{01}^2}\gamma_z e^{j[kr_{01} - \tan^{-1}(kr_{01})]}$$

where $\gamma_z = \frac{z_0}{r_{01}}$

For $r_{01} >> \lambda$ and $\gamma_z \approx 1$,
it has Spherical wavefront

2.2 Fresnel wavelet

With approximation $r_{01} \gg \lambda$ and $\frac{1}{r_{01}^2} \approx \frac{1}{z_0^2}$
exp(−jφ) goes to the limiting value of −j.

Then, we get the Fresnel wavelet

$$h(P_0;P_1) = \frac{-je^{jkz_0}}{\lambda z_0}\exp\left\{\frac{jk}{2z_0}\left[(x_0-x_1)^2+(y_0-y_1)^2\right]\right\}$$

,which has Parabolic wavefront

The Fresnel diffraction fomula is

$$U(x_0,y_0)$$

$$= \frac{je^{jkz0}}{\lambda z_0}\exp\left[j\frac{k}{2z_0}(x_0^2+y_0^2)\right]\mathbf{F}_\eta\mathbf{F}_\xi\left\{U_{z=0}(x_1,y_1)\exp\left[\frac{jk}{2z_0}(x_1^2+y_1^2)\right]\right\}$$

where $\xi = x_0/(\lambda z_0)$, $\eta = y_0/(\lambda z_0)$

This approximation is valid when

$$z_0 \gg \sqrt[3]{\frac{\pi}{4}}\lambda\left[\left(\frac{x_0-x_1}{\lambda}\right)^2 + \left(\frac{y_0-y_1}{\lambda}\right)^2\right]_{\max}^{2/3}$$

However, the Fresnel approximation is valid over a larger range than the above condition, due to the “Principal of Stationary Phase.”

2.3 Fraunhofer wavelet

With approximation $z_0 \gg \frac{k}{2}(x_1^2+y_1^2)_{\max}$

Then, we get the Fraunhofer wavelet

$$h(P_0;P_1) = \frac{je^{jkz_0}}{\lambda z_0}e^{j\frac{k}{2z_0}(x_0^2+y_0^2)}\exp\left[\frac{2\pi j}{\lambda z_0}(x_0x_1+y_0y_1)\right]$$

,which has Planar wavefront.

The Fraunhofer diffraction fomula is

$$U(x_0,y_0)$$

$$= \frac{je^{jkz_0}}{\lambda z_0}e^{j\frac{k}{2z_0}(x_0^2+y_0^2)}\mathbf{F}_{\eta=\frac{y_0}{\lambda z_0}}\mathbf{F}_{\xi=\frac{x_0}{\lambda z_0}}[U_{z=0}(x_1,y_1)]$$

In fact, at the observation plane, direction cosines

$$\alpha = \lambda\xi$$
$$\beta = \lambda\eta$$
$$\gamma = \sqrt{1-\alpha^2-\beta^2}$$
$$= \sqrt{1-(\lambda\xi)^2-(\lambda\eta)^2},$$

determine the angular orientation of each plane wave.

2.4 Diffraction using Transfer function

Using the “exact” Huygens wavelet”, the transfer function of free space is

$$\mathbf{F}_\eta\mathbf{F}_\xi[h(P_0;P_1)] = e^{jk\gamma z_0} = H(\gamma;z_0)$$

The diffraction fomula is

$$\mathbf{F}_\eta\mathbf{F}_\xi[U(P_0)] = \mathbf{F}_\eta\mathbf{F}_\xi[U_{z=0}(P_1)]e^{jk\gamma z_0}$$

Also, we can define “spatial frequency spectrum”(or “Angular spectrum of plane waves”)

$$\mathbf{F}_\eta\mathbf{F}_\xi[U_{z=0}(P_1)] = \iint_\infty U_{z=0}(x_1,y_1)e^{-j2\pi(\xi x_1+\eta y_1)}dx_1dy_1$$
$$= A_{z=0}(\xi,\eta).$$

For an example, given

$$U_{z=0}(x_1) = \cos[2\pi(\alpha_0/\lambda)x_1] = \cos(2\pi\xi_0x_1)$$

The “Angular spectrum of plane waves” is

$$A_{z=0}(\xi,\eta) = \iint_\infty \cos(2\pi\xi_0x_1)e^{-j2\pi(\xi x_1+\eta y_1)}dx_1dy_1$$

$$= \frac{1}{2}\left[\delta(\xi-\xi_0,\eta)+\delta(\xi+\xi_0,\eta)\right].$$

The field at the observation plane is

$$U(x_0,y_0;z_0)$$

$$= \mathbf{F}_{y_0}^{-1}\mathbf{F}_{x_0}^{-1}[A_{z=0}(\xi,\eta)e^{jk\gamma z_0}]$$

$$= \mathbf{F}_{y_0}^{-1}\mathbf{F}_{x_0}^{-1}\left\{\frac{1}{2}\left[\delta(\xi-\xi_0,\eta)+\delta(\xi+\xi_0,\eta)\right]e^{jk\gamma z_0}\right\}$$

$$= \frac{1}{2}\mathbf{F}_{y_0}^{-1}\mathbf{F}_{x_0}^{-1}\left[e^{jkz_0\sqrt{1-(\lambda\xi)^2-(\lambda\eta)^2}}\delta(\xi-\xi_0,\eta) + e^{jkz_0\sqrt{1-(\lambda\xi)^2-(\lambda\eta)^2}}\delta(\xi+\xi_0,\eta)\right]$$

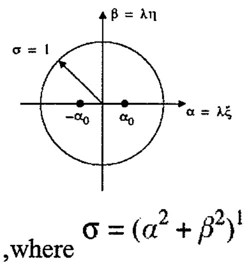
$$= \frac{1}{2}\left[e^{jkz_0\sqrt{1-(\lambda\xi_0)^2}}e^{j2\pi\xi_0x_0} + e^{jkz_0\sqrt{1-(\lambda\xi_0)^2}}e^{-j2\pi\xi_0x_0}\right]$$

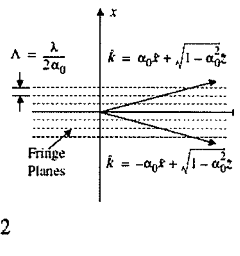
$$= \frac{1}{2}\left[e^{j2\pi\xi_0x_0} + e^{-j2\pi\xi_0x_0}\right]e^{jkz_0\sqrt{1-(\lambda\xi_0)^2}}$$

$$= \cos(2\pi\xi_0x_0)e^{jkz_0\sqrt{1-(\lambda\xi_0)^2}}$$

$$= \cos(k\alpha_0x_0)e^{jkz_0\sqrt{1-\alpha_0^2}}$$

We can think this result as combination of two plane waves





,where $\sigma = (\alpha^2+\beta^2)^{1/2}$

$|\sigma| \leq 1$: Propagating component

$|\sigma| > 1$: evanescent components

III. Important Diffraction Topics

3.1 Talbot Effect

If the period of object is >> wavelength

The maximum extent of the object's angular spectrum is limited to small angles: $\sigma \ll 1$

For the Fraunhofer region, the free space transfer function H becomes

$$e^{jkz_0\gamma} \approx e^{jkz_0} e^{-j\frac{kz_0}{2}\alpha^2} = e^{jkz_0} e^{-j\frac{kz_0}{2}(\xi\lambda)^2}$$

For a given weak phase grating,

$$U_{z=0}(x_1) = e^{jm\cos(2\pi x_1/T)}$$
$$\approx 1 + jm\cos(2\pi x_1/T)$$

The angular spectrum at z=0 is

$$A_{z=0}(\xi, \eta)$$
$$= \iint_{-\infty}^{\infty} [1 + jm\cos(2\pi x_1/T)] e^{-j2\pi(\xi x_1 + \eta y_1)} dx_1 dy_1$$
$$= \left\{\delta(\xi) + \frac{1}{2}jm\left[\delta(\xi - \xi_0) + \delta(\xi + \xi_0)\right]\right\}\delta(\eta) ,$$

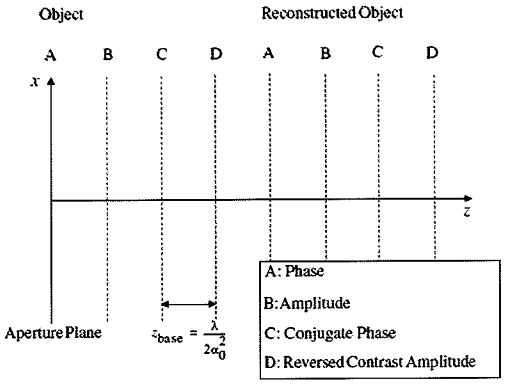
The angular spectrum at the observation plane is

$$A(\xi, \eta; z_0)$$
$$= e^{jkz_0}\left\{\delta(\xi) + \frac{1}{2}jme^{-j\frac{kz_0}{2}(\lambda\xi_0)^2}\left[\delta(\xi - \xi_0) + \delta(\xi + \xi_0)\right]\right\}\delta(\eta)$$
$$= e^{jkz_0}\left\{\delta(\xi) + \frac{1}{2}jme^{j\psi}\left[\delta(\xi - \xi_0) + \delta(\xi + \xi_0)\right]\right\}\delta(\eta) .$$

It really depends on

$$\psi = -(kz_0/2)(\lambda\xi)^2$$

,and the propagated field is repeated as



$$z_{base} = \frac{1}{2\lambda\xi_0^2} = \frac{\lambda}{2\alpha_0^2}$$

,where

3.2 Babinet's Principle

Since diffraction is a linear and shift invariant process,

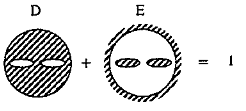
$$\mathbf{L}(U_{total}) = \mathbf{L}(U_1) + \mathbf{L}(U_2) + \ldots + \mathbf{L}(U_n)$$

,where L: scalar diffraction linear operation

U_{total} : total field transmitted through aperture

$U_1, U_2, \ldots U_n$: Divided sub apertures

Then, the aperture algebra is very useful.

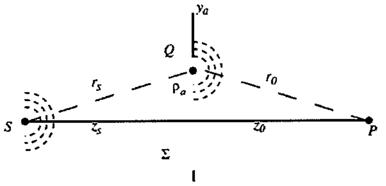


example:

Also, D and E is complementary apertures in this case.

3.3 Fresnel Zones

Given aperture and source and observation positions as below



$$\rho_a = \sqrt{x_a^2 + y_a^2}, \quad L = \frac{z_s z_0}{z_s + z_0}$$,a:aperture radi

The Fresnel number is

$$N_f = \frac{a^2}{\lambda L}$$

The radius of the m^{th} Fresnel zone is given by

$$\rho_m = \sqrt{m\lambda L}$$

the area of each Fresnel zone is

$$A_f = \pi\rho_{m+1}^2 - \pi\rho_m^2 = \pi\lambda L$$

3.4 Diffraction behind circular aperture

Using the Fresnel zones, the axial field behind the circular aperture is

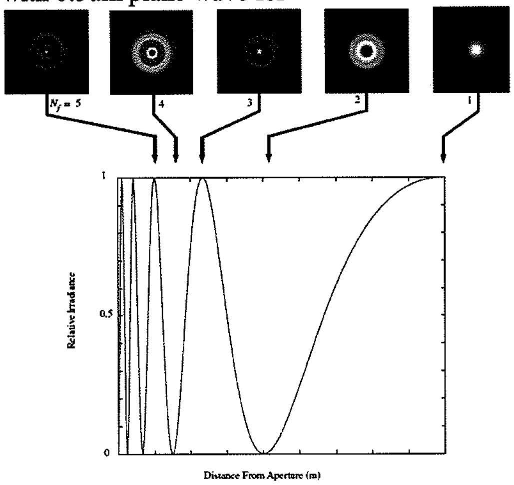
$$U(P) = -U_\infty(P)(e^{j\pi N_f} - 1)$$

,where $U_\infty(P)$: field w/o the aperture

Then, the axial irradiance becomes

$$I(P) = U(P)U^*(P)$$
$$= 4I_\infty(P)\sin^2\left(\frac{\pi N_f}{2}\right)$$

Case 1) 1mm diameter aperture, illuminated with 0.5um plane wave for $N_f = 1$ through 5.



case 2) converging wave illumination

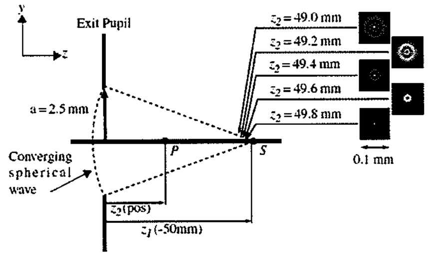
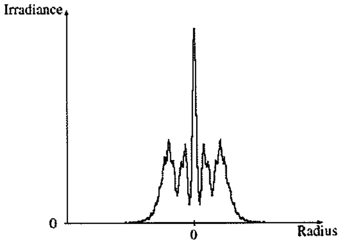


Fig. 6.36. Geometry for a laser beam converging through an aperture. $\lambda = 0.5\mu m$, $f = 50mm$ and $a = 2.5mm$.

The transverse irradiance also can be calculated. It has N_f (=5) peaks as below.



3.5 Poisson's Spot

Behind a circular disk, on-axis filed is

$$U_{disk}(P) = U_\infty(P) - U_{aperture}(P)$$
$$= U_\infty(P) - U_\infty(P)(1 - e^{j\pi N_f})$$
$$= U_\infty(P)e^{j\pi N_f},$$

Thus,

$$I_{disk}(P) = I_\infty(P)$$

3.6 Fresnel Zone Plate

If we put mask on even or odd Fresnel zones, it become the binary amplitude Fresnel Zone Plate. The on-axis irradiance becomes

$$I(P) = 4N_{FZ}^2 I_\infty(P)$$

,where N_{FZ} : number of open Fresnel zones

The transverse irradiance also can be calculated. It is has approximate width of the central spot

$$w = \frac{\lambda z_0}{a}$$

, which is close to the ideal lens case

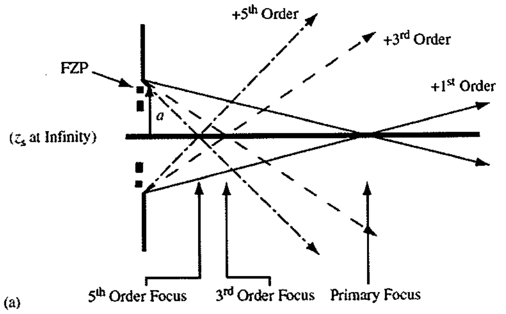
$$w = 1.22\frac{\lambda z_0}{a}$$

, where w: width between zeros of the central spot in Airy disk

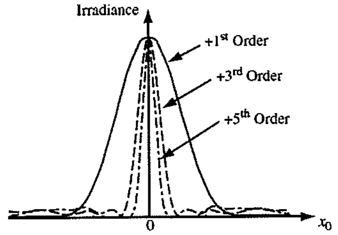
This Fresnel zone plate is very close to a lens, which has multiple focal lengths as

$$f_n = \frac{a_{ZP1}^2}{n\lambda}$$

,where n = ...-5, -3, -1, 1, 3, 5 ...

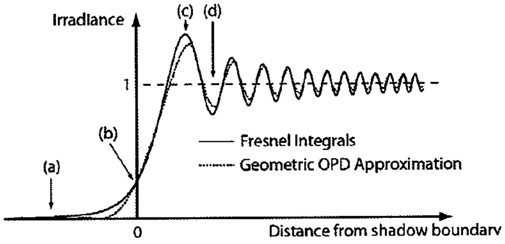


The transverse irradiance is shown as



3.7 Edge diffraction

The diffraction from an edge has irradiance



3.8 Fraunhofer diffraction

Case 1) Circular aperture w/ normal incident plane wave

$$U(\rho_0)=-\frac{je^{jkz_0}}{\lambda z_0}\exp\left(j\frac{k\rho_0^2}{2z_0}\right)\mathbf{B}_{\sigma_0=\frac{\rho_0}{\lambda z_0}}\left[\operatorname{circ}\left(\frac{\rho_1}{2a}\right)\right]$$

$$=-\frac{je^{jkz_0}}{\lambda z_0}\exp\left(j\frac{k\rho_0^2}{2z_0}\right)\pi a^2\operatorname{somb}\left(\frac{2a\rho_0}{\lambda z_0}\right)$$

Then, the irradiance becomes

$$I(\rho_0)=\left(\frac{1}{\lambda z_0}\right)^2\left(\pi a^2\right)^2\operatorname{somb}^2\left(\frac{2a\rho_0}{\lambda z_0}\right)$$

We can also specify minimum resolvable distance

1) by Rayleigh's criterion

$$\rho_{RES}=1.22\frac{\lambda z_0}{2a}=0.61\frac{\lambda}{NA}=1.22\lambda f/\#$$

2) by Sparrow criterion

$$\rho_{RES-SPARROW}=0.94\frac{\lambda z_0}{2a}=0.47\frac{\lambda}{NA}=0.94\lambda f/\#$$

Case 2) Rectangular aperture w/ normal incident plane wave

$$U(x_0,y_0)=$$

$$=-\frac{je^{jkz_0}}{\lambda z_0}\exp\left[\frac{k}{2z_0}\left(x_0^2+y_0^2\right)\right]\mathbf{F}_{\eta=\frac{y_0}{\lambda z_0}}\mathbf{F}_{\xi=\frac{x_0}{\lambda z_0}}\left[A\operatorname{rect}\left(\frac{x_1}{2a}\right)\operatorname{rect}\left(\frac{y_1}{2b}\right)\right]$$

$$=-\frac{jAe^{jkz_0}}{\lambda z_0}\exp\left[\frac{k}{2z_0}\left(x_0^2+y_0^2\right)\right]4ab\operatorname{sinc}\left(\frac{4ax_0}{\lambda z_0}\right)\operatorname{sinc}\left(\frac{4by_0}{\lambda z_0}\right).$$

Then, the irradiance becomes

$$I(x_0,y_0)=\left(\frac{A}{\lambda z_0}\right)^2(4ab)^2\operatorname{sinc}^2\left(\frac{2ax_0}{\lambda z_0}\right)\operatorname{sinc}^2\left(\frac{2by_0}{\lambda z_0}\right)$$

The width between zeros of the central spot

$$s_x=\frac{\lambda z_0}{a}=\frac{\lambda}{NA_x}\text{, and }\quad s_y=\frac{\lambda z_0}{b}=\frac{\lambda}{NA_y}$$

Case 3) Double slit w/ normal incident plane wave

$$U(0,y_0)=$$

$$=-\frac{je^{jkz_0}}{\lambda z_0}\exp\left(\frac{ky_0^2}{2z_0}\right)\mathbf{F}_{\eta=\frac{y_0}{\lambda z_0}}\mathbf{F}_{\xi=\frac{x_0}{\lambda z_0}}\left\{A\operatorname{rect}\left(\frac{x_1}{2a}\right)\operatorname{rect}\left(\frac{y_1}{2b}\right)\right.$$

$$\left.\left.*\left[\delta(y_1+d/2)+\delta(y_1-d/2)\right]\right\}\right|_{x_0=0}$$

$$=-\frac{jAe^{jkz_0}}{\lambda z_0}\exp\left(\frac{ky_0^2}{2z_0}\right)4ab\operatorname{sinc}\left(\frac{2by_0}{\lambda z_0}\right)2\cos\left(\frac{2\pi dy_0}{2\lambda z_0}\right)$$

Then, the irradiance becomes

$$I(0,y_0)=4\left(\frac{A}{\lambda z_0}\right)^2(4ab)^2\operatorname{sinc}^2\left(\frac{2by_0}{\lambda z_0}\right)\cos^2\left(\frac{2\pi dy_0}{2\lambda z_0}\right)$$

$$=4\left(\frac{A}{\lambda z_0}\right)^2(4ab)^2\operatorname{sinc}^2\left(\frac{2by_0}{\lambda z_0}\right)\left[\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi dy_0}{\lambda z_0}\right)\right]$$

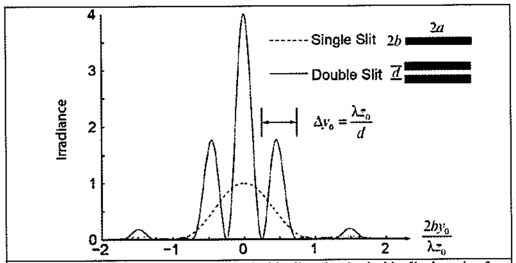


Figure 6.61. Slit diffraction for single and double slits. For the double slit, the ratio of separation to slit width $d/2b = 2$, or $d/b = 4$.

Case 4) N slits (or grating)

For N slits, the number of secondary peaks between the primary peaks equals N-2

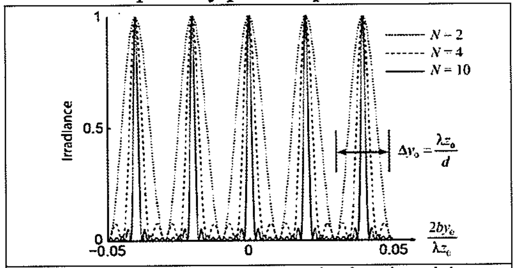


Figure 6.63. Increasing the number of slits N. The number of secondary peaks between the primary peaks equals N-2. In this figure, the peaks are normalized to unity irradiance. In reality, the peak irradiance increases as N^2. Also, $d/b = 100$.

For gratings, it follows the grating equation

$$n_S\sin\theta_S+n_0\sin\theta_m=\frac{m\lambda}{d}$$

For a given transmission of aperture

$$t(x_1,y_1)=\left[f_d(y_1)*\frac{1}{d}\operatorname{comb}\left(\frac{y_1}{d}\right)\right]\frac{1}{2a}\operatorname{circ}\left(\frac{\sqrt{x_1^2+y_1^2}}{2a}\right)$$

and given plane wave illumination

$$U(x_1,y_1)=A\exp\left(j2\pi\frac{\beta_S}{\lambda}y_1\right)$$

The irradiances is (for 2a >> d)

$$I(x_0,y_0)\approx$$

$$\left(\frac{\pi a^2A}{\lambda z_0 d}\right)^2\sum_{m=-\infty}^{\infty}\left|\mathbf{F}_{\frac{m}{d}}\left[f_d(y_1)\right]\right|^2\operatorname{somb}^2\left[\frac{2a}{\lambda z_0}\sqrt{x_0^2+\left(y_0+\frac{m\lambda z_0}{d}-\beta_S z_0\right)^2}\right]$$

and the absolute diffraction efficiency is

$$E_m=\left|\frac{1}{d}\mathbf{F}_{\frac{m}{d}}\left[f_d(y_1)\right]\right|^2$$

For example, consider a sine wave grating with

$$f_d(y_1)=\left[\frac{1}{2}+\frac{1}{2}\sin(2\pi y_1/d)\right]\operatorname{rect}\left(\frac{y_1}{d}\right)$$

so

$$E_m=\left|\frac{1}{d}\mathbf{F}_{\frac{m}{d}}\left\{\left[\frac{1}{2}+\frac{1}{2}\sin(2\pi y_1/d)\right]\operatorname{rect}\left(\frac{y_1}{d}\right)\right\}\right|^2$$

$$=\left|\frac{1}{2}\operatorname{sinc}(m)+\frac{\operatorname{sinc}(m-1)-\operatorname{sinc}(m+1)}{4j}\right|^2$$

E_0 = 0.25 (25%) and $E_{+/-1}$ = 0.0625 (6.25%)

case 5) Wide-angle diffraction to a spherical surface

Gratings often diffract light over large angles, which are not strictly allowed in the Fraunhofer development in previous section. This restrictions are removed by developing a diffraction formula for propagating from a plane to a spherical surface.

For a observation surface, which is a sphere of radius r, and centered at (x1,y1)=(0,0), the field is

$$U(\xi,\eta)=-\frac{j\gamma e^{jkr}}{\lambda r}\mathbf{F}_{\frac{\beta}{\lambda}=\eta}\mathbf{F}_{\frac{\alpha}{\lambda}=\xi}\left[U_{z=0}(x_1,y_1)\right]$$

The observation coordinate system is defined on a hemisphere

$$x_0=\lambda\xi r$$

$$y_0=\lambda\eta r$$

, where $z_0=\gamma r$

case 6) Conical diffraction

Illumination plane wave propagation vector is not in the plane defined by g (grating vector) and the normal to the grating surface.

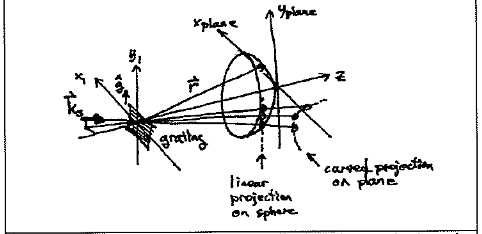


Figure 6.72. Conical diffraction. Projection of diffraction orders onto the observation hemisphere and the flat vertex plane.

The conical nature is a result of observing the diffraction on a plane, rather than a sphere. The diffraction orders are said to be linear in "direction cosine space."