Solutions to the Wave Equation (Part B)

Diffraction and Interferometry

Polarization

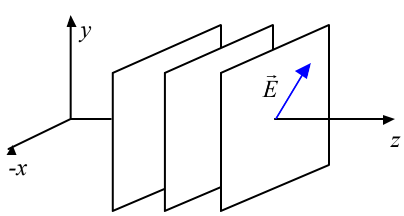
Consider the case of an EM plane wave traveling in the +*z* direction.

E oscillates perpendicular to *z*.

General solution:

$$\vec{E}(z,t) = \vec{E}_0 e^{j(kz-\omega t)} = \left[A_x\hat{x} + A_y e^{j\phi}\hat{y} \right] e^{j(kz-\omega t)}$$

We will now trace the tip of the electric vector $\vec{E}$ for several special cases.



We will look at both:

fixed $z = z_0$ with $t =$ variable

fixed $t = t_0$ with $z =$ variable

Polarization – Linear

$A_x = A_y = A_0$
$\phi = 0$

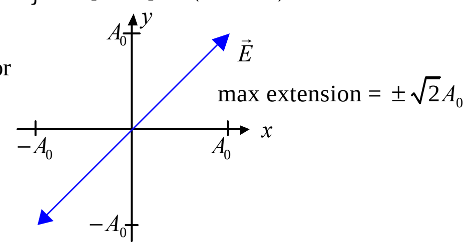
$$\vec{E}(z,t) = \vec{E}_0 e^{j(kz-\omega t)} = A_0\left[\hat{x}+\hat{y}\right]e^{j(kz-\omega t)}$$

$$\text{Re}\left\{\vec{E}(z,t)\right\} = A_0\left[\hat{x}+\hat{y}\right]\cos\left(kz-\omega t\right)$$ ← This is the physical wave.

Consider $z = z_0$, where $\text{Re}\left\{\vec{E}(z_0,t)\right\} = A_0\left[\hat{x}+\hat{y}\right]\cos\left(kz_0-\omega t\right)$

Trace the tip of the electric vector in the (x,y) plane in time:

This is true for *all* z_0.



Polarization – Left Circular

$A_x = A_y = A_0$
$\phi = \pi/2$

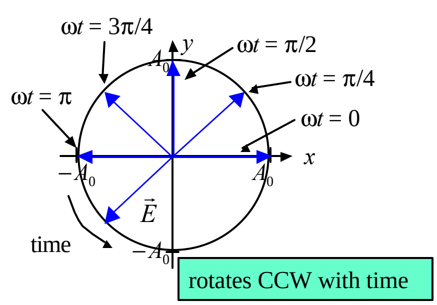
$$\vec{E}(z,t) = A_0\left[\hat{x}+e^{j\pi/2}\hat{y}\right]e^{j(kz-\omega t)}$$

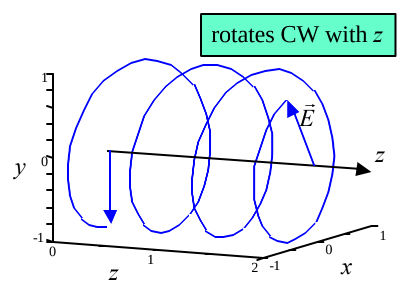
Consider $z = 0$, where

$$\text{Re}\left\{\vec{E}(0,t)\right\} = A_0\left[\cos(\omega t)\hat{x}+\cos(\pi/2-\omega t)\hat{y}\right]$$
$$= A_0\left[\cos(\omega t)\hat{x}+\sin(\omega t)\hat{y}\right]$$

Consider $t = 0$, where

$$\text{Re}\left\{\vec{E}(z,0)\right\} = A_0\left[\cos(kz)\hat{x}+\cos(\pi/2+kz)\hat{y}\right]$$
$$= A_0\left[\cos(kz)\hat{x}-\sin(kz)\hat{y}\right]$$





Polarization – Right Circular

$A_x = A_y = A_0$
$\phi = -\pi/2$

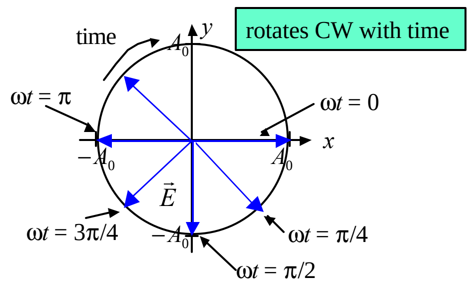
$$\vec{E}(z,t) = A_0\left[\hat{x} + e^{-j\pi/2}\hat{y}\right]e^{j(kz-\omega t)}$$

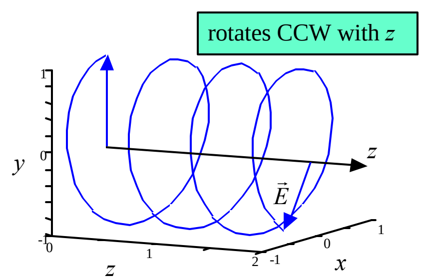
Consider $z = 0$, where

$$\mathrm{Re}\left\{\vec{E}(0,t)\right\} = A_0\left[\cos(\omega t)\hat{x} + \cos(-\pi/2 - \omega t)\hat{y}\right] = A_0\left[\cos(\omega t)\hat{x} - \sin(\omega t)\hat{y}\right]$$

Consider $t = 0$, where

$$\mathrm{Re}\left\{\vec{E}(z,0)\right\} = A_0\left[\cos(kz)\hat{x} + \cos(-\pi/2 + kz)\hat{y}\right] = A_0\left[\cos(kz)\hat{x} + \sin(kz)\hat{y}\right]$$





Polarization – Elliptical

A_x , A_y
ϕ_0

$$\vec{E}(z,t) = \left[A_x\hat{x} + A_y e^{j\phi}\hat{y}\right]e^{j(kz-\omega t)}$$,where

$$E_x = A_x \cos(kz - \omega t) \quad (1)$$

$$E_y = A_y \cos(kz - \omega t + \phi) \quad (2)$$

Combination of (1) and (2) gives:

$$\left(\frac{E_x}{A_x}\right)^2 + \left(\frac{E_y}{A_y}\right)^2 - 2\left(\frac{E_x}{A_x}\right)\left(\frac{E_y}{A_y}\right)\cos\phi = \sin^2\phi$$,

which is an equation of an ellipse.

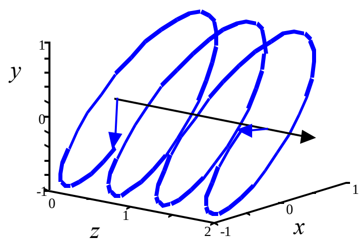
In other words,

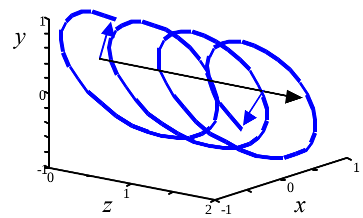
The tip of the electric vector specified by E_x and E_y trace an ellipse in any plane z that is determined by A_x, A_y and ϕ.

Polarization – Elliptical

$A_x = A_y = 1$
$\phi = \pi / 4$

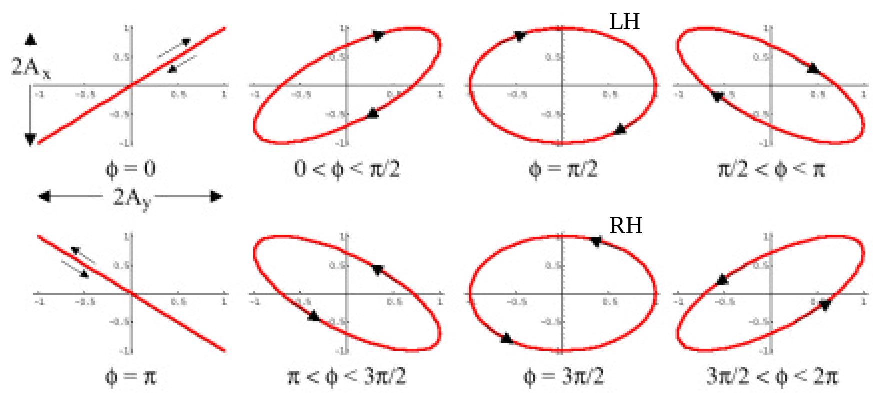
$A_x = A_y = 1$
$\phi = 5\pi / 4$





Polarization – Elliptical

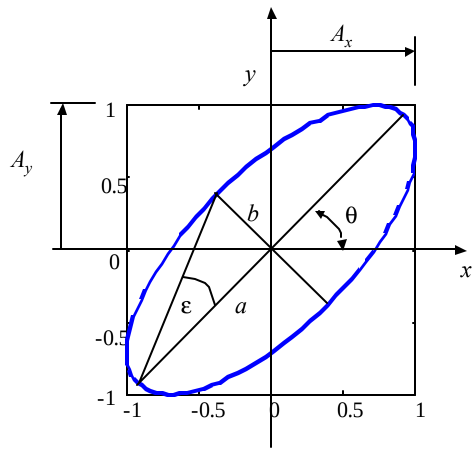
(Arrows show trace in *z* direction)



This slide from Jim Wyant, 2000.

Polarization – Ellipticity

Look into wave and observe the trace at z_0 as a function of time



$$\tan\alpha = \frac{A_y}{A_x} \quad (0 \le \alpha \le \pi/2)$$

$$\tan\varepsilon = \pm\frac{b}{a} \quad (-\pi/4 \le \varepsilon \le -\pi/4)$$

$$\tan 2\theta = (\tan 2\alpha)\cos\phi \quad (0 \le \theta \le \pi)$$

$$\sin 2\varepsilon = (\sin 2\alpha)\sin\phi$$

ε is often called the *ellipticity*

Polarization – Jones* Calculus

Look into wave and observe the trace at z_0 as a function of time

The values of constants A_x, A_y and ϕ determine the *state of polarization* of the plane wave. Various states include *linear*, *circular* and *elliptical.*

We can write the wave conveniently with these constants in a column vector:

$$\vec{E}(x,t) = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} A_x \\ A_y e^{j\phi} \end{pmatrix} e^{j(kz-\omega t)} \quad .$$

As the wave propagates, the state of polarization does not change unless the wave encounters and optical element. The *Jones vector* is the state of polarization written as as column vector.

$$\text{Jones vector} = J = \begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} A_x \\ A_y e^{j\phi} \end{pmatrix}$$

*R. Clark Jones, *JOSA A,* 31, 488-493 (1941), provided on web site. Actually, there are a series of eight papers.

Polarization – Jones Vector Examples

Common constants may be divided out of each Jones vector component.

Example: $J = \begin{pmatrix} 3e^{j\pi/2} \\ 9e^{j\pi/2} \end{pmatrix} = 3e^{j\pi/2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

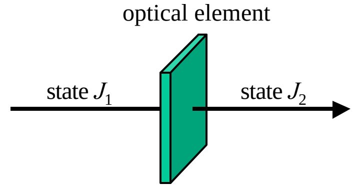
Various states of polarization:

linear x	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	linear y	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
linear +45°	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	linear -45°	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
RHC	$\begin{pmatrix} 1 \\ -j \end{pmatrix}$	LHC	$\begin{pmatrix} 1 \\ j \end{pmatrix}$

Polarization – Jones Matrices for Optical Elements

Optical elements can affect the state of polarization as a plane wave passes through them.

optical element

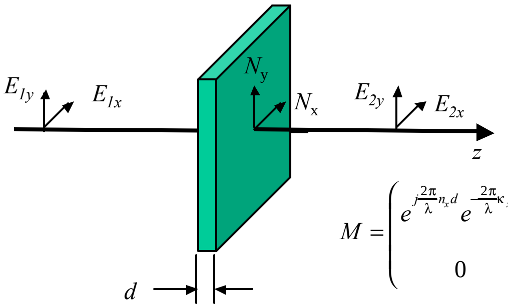


If we assume that the element acts linearly, the change of state through an element can be represented by a 2-by-2 matrix *M*.

$$J_2 = MJ_1 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} J_1 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} J_{1x} \\ J_{1y} \end{pmatrix} = \begin{pmatrix} m_{11}J_{1x} + m_{12}J_{1y} \\ m_{21}J_{1x} + m_{22}J_{1y} \end{pmatrix}$$

Polarization – Jones Matrices for Optical Elements

Consider optical elements with polarization-dependent refractive index.



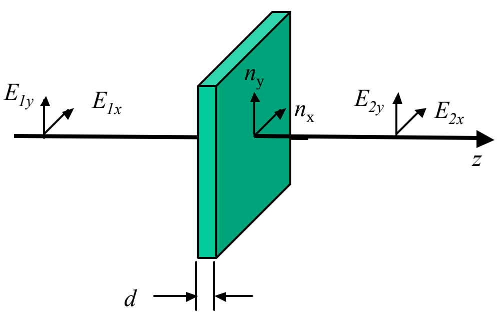
$$N_x = n_x + j\kappa_x$$
$$N_y = n_y + j\kappa_y$$

$$M = \begin{pmatrix} e^{j\frac{2\pi}{\lambda}n_x d} e^{-\frac{2\pi}{\lambda}\kappa_x d} & 0 \\ 0 & e^{j\frac{2\pi}{\lambda}n_y d} e^{-\frac{2\pi}{\lambda}\kappa_y d} \end{pmatrix}$$

Retardation plate: $n_x \neq n_y$, $\kappa_x \approx \kappa_y \approx 0$
Polarizer plate: $n_x \approx n_y$, $\kappa_x \neq \kappa_y$

Polarization – Jones Matrix Examples

Uniaxial Crystals



$n_x = n_o$
$n_y = n_e$ (OA)
$n_z = n_o$

Uniaxial crystals have two primary refractive indices, depending on how the light beam is polarized. n_o (for *ordinary*) and n_e (for *extraordinary*). A common configuration is shown above. n_e defines the *optical axis* OA. A *positive* crystal, like quartz, has $n_e > n_o$. A *negative* crystal, like calcite, has $n_o > n_e$.

Polarization – Jones Matrix Examples

Quarter-Wave Plate

A quartz quarter-wave plate is a uniaxial crystal with

$$n_0 = n_x = 1.544$$
$$n_e = n_y = 1.553$$
$$\kappa_e = \kappa_o \approx 0$$
$$\text{at } \lambda = 500 \text{ nm}$$

The thickness d is fabricated so that the phase difference between x and y polarization components through the plate is $\pi/2$.

$$J_2 = c\begin{pmatrix} 1 & 0 \\ 0 & e^{j\pi/2} \end{pmatrix} J_1 = M_{QWP} J_1 = \begin{pmatrix} e^{j\frac{2\pi}{\lambda}n_o d} & 0 \\ 0 & e^{j\frac{2\pi}{\lambda}n_e d} \end{pmatrix} J_1 \quad ,$$

where c is a constant. So,

$$c\begin{pmatrix} 1 & 0 \\ 0 & e^{j\pi/2} \end{pmatrix} = \begin{pmatrix} e^{j\frac{2\pi}{\lambda}n_o d} & 0 \\ 0 & e^{j\frac{2\pi}{\lambda}n_e d} \end{pmatrix} = e^{j\frac{2\pi}{\lambda}n_o d} \begin{pmatrix} 1 & 0 \\ 0 & e^{j\frac{2\pi}{\lambda}(n_e - n_o)d} \end{pmatrix} \quad .$$

Polarization – Jones Matrix Examples

Quarter-Wave Plate

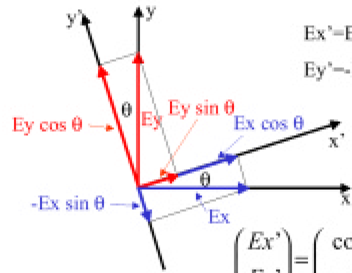
Comparison of terms yields

$$\frac{\pi}{2} = \frac{2\pi}{\lambda}(n_e - n_o)d, \text{ or}$$

$$d = \frac{\lambda}{4}\frac{1}{n_e - n_o} = \frac{500 \text{ nm}}{4(1.553\text{-}1.544)} = 14 \times 10^3 \text{ nm} = 14 \mu\text{m}$$

Polarization – Rotated Elements

In order to describe the effect of a rotated optical element, we can project the (x,y) electric fields onto the element with the *rotation matrix*, M_R.



Ex'=Ex cos θ + Ey sin θ

Ey'=-Ex sin θ + Ey cos θ

$$\begin{pmatrix} Ex' \\ Ey' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} Ex \\ Ey \end{pmatrix}$$

$$M_R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Part of this slide from Jim Wyant, 2000.

Polarization – Jones Matrix Examples

Rotated Quarter-Wave Plate

Consider rotating a quarter-wave plate by 45°. Follow these steps to find the effect on transmitted light:

1.) Decompose the incident light into components. Application of the rotation matrix produces a description of the light field *in the coordinate frame of the rotated plate.*

$$J_1' = M_R J_1$$

2.) Apply the quarter-wave plate matrix.

$$J_2' = M_{QWP} J_1' = M_{QWP} M_R J_1$$

3.) Rotate back into the (x,y) coordinate system.

$$J_2 = M_R^\dagger J_2' = M_R^\dagger M_{QWP} J_1' = M_R^\dagger M_{QWP} M_R J_1 = M_{QWP}^R J_1$$

Notice that

$$M_{QWP}^R = M_R^\dagger M_{QWP} M_R \qquad (\dagger = \text{transpose})$$

Polarization – Jones Matrix Examples

Rotated Quarter-Wave Plate (cont'd)

$$M_{QWP}^{R} = M_R^{\dagger} M_{QWP} M_R$$

$$= \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} = \frac{1+i}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

A quarter-wave plate rotated at 45° has a very simple equivalent matrix.

Polarization – Jones Matrix Examples

Rotated Quarter-Wave Plate (cont'd)

Look at the effect of a rotated QWP on *x*-polarized light.

$$J_2 = M_{QWP}^{R} J_1$$

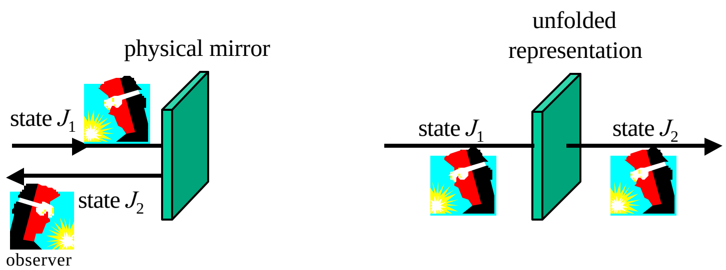
$$= \frac{1+i}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1+i}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

The effect of a QWP (rotated at 45°) on *x*-polarized light is to change the state of polarization to RHC.

Polarization – Jones Matrix Examples

Reflection Off a Mirror

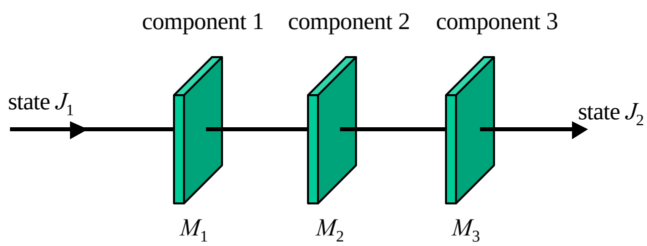


Remember, we look *into* the wave to determine the state of polarization. A π phase is introduced between x and y axes from the physical mirror due to the right-handed coordinate system used in the wave description. Therefore, the representation of a perfect mirror in the *unfolded* system is:

$$M_M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Polarization – Jones Matrix Systems

Cascaded Components

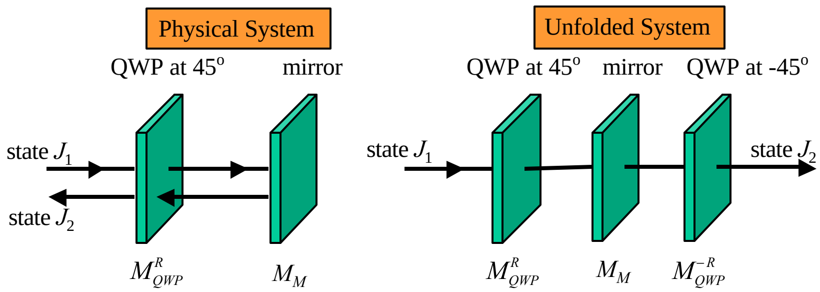


$$J_2 = M_3 M_2 M_1 J_1 = M_{SYS} J_1$$

Cascaded elements can be represented by algebraically multiplying component matrices into a system matrix M_{SYS}.

Polarization – Jones Matrix Examples

Combination of a Quarter-Wave Plate and a Mirror



$$J_2 = M_{QWP}^{-R} M_M M_{QWP}^{R} J_1$$

$$M_M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad M_{QWP}^{R} = \frac{1+i}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$M_{QWP}^{-R} = \frac{1+i}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

Polarization – Jones Matrix Examples

Combination of a Quarter-Wave Plate and a Mirror (cont'd)

$$J_2 = \frac{1+i}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1+i}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} J_1$$
$$= \frac{i}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} J_1$$
$$= \frac{i}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} J_1$$
$$= \frac{i}{2}\begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} J_1$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} J_1$$

Notice the effect on *x*-polarized light:

If $J_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The state of polarization is now *y* polarized.

Polarization – Jones Matrix Examples

Linear polarizer *x*	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Linear polarizer *y*	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
HWP	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
General retarder	$e^{j\phi/2}\begin{pmatrix} 1 & 0 \\ 0 & e^{j\phi} \end{pmatrix}$