Super Tonks-Girardeau states of trapped 1D Bose and Fermi gases

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Plan

- FB mapping method for strongly correlated ultracold gases

  History (TG gas), experimental realization, FB mapping

- Bosonic sTG gas: metastability against collapse despite strong attractions

  Theoretical prediction, recent observation, explanation via FB mapping

- TWO sTG phases of spinor fermions

  Bose sTG gas of fermion dimers vs. sTG-ideal Fermi gas hybrid

TG gas history

- First treatment of 1D hard-sphere gas: L. Tonks, Phys. Rev. **50**, 955 (1936): *Classical, high temperature, inapplicable to QM of ultracold atomic vapors*

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  T. Nagamiya, Proc. Phys. Math. Soc. Japan **22**, 705 (1940)

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  T. Nagamiya, Proc. Phys. Math. Soc. Japan **22**, 705 (1940)
- Later independent derivations:
  H. Stachowiak, Acta Univ. Wratislaviensis **12**, 93 (1960)

  M. Girardeau, J. Math. Phys. **1**, 516 (1960) *Fermi-Bose mapping: Both ground and excited states, many generalizations*

Strongly interacting 1D atomic gases

- Bosons : the Tonks-Girardeau (TG) gas $\equiv$ 1D Bose gas with repulsive interactions $g^B_{1D}\delta(x_1 - x_2)$ with $\gamma^B_{1D} \longrightarrow \infty$
  Remark: $\gamma^B_{1D} = 2mg^B_{1D}/\hbar^2 n \Rightarrow$ the TG limit is reached at *low density* or *large effective mass*

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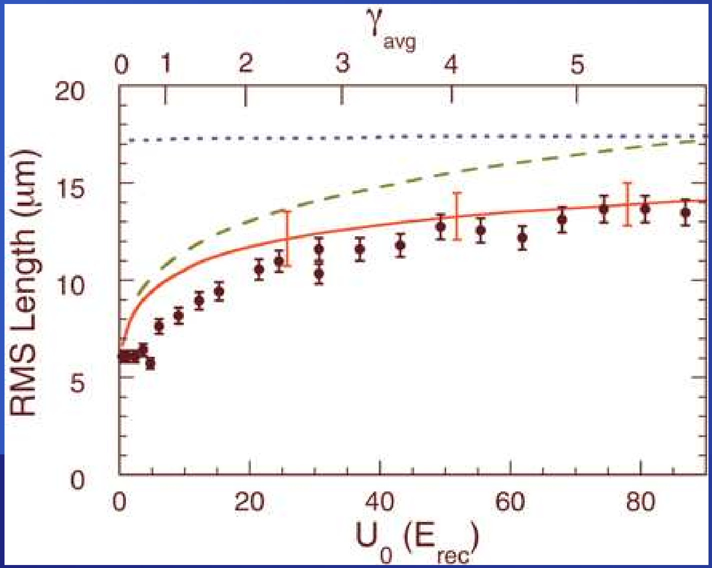
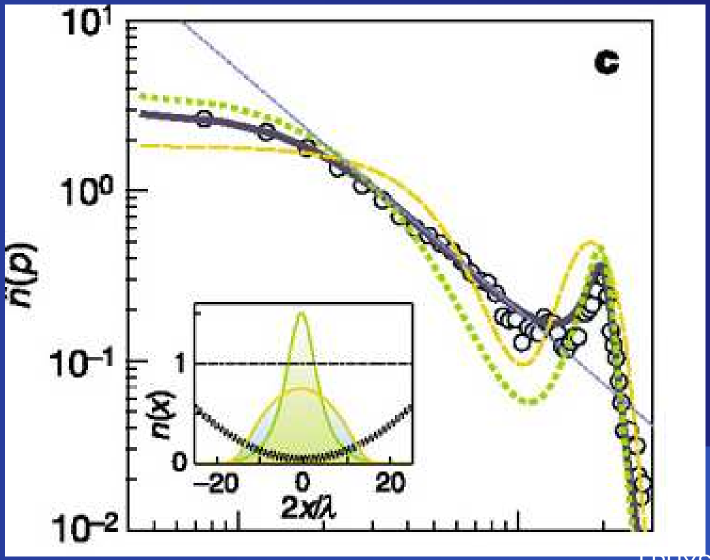
- Experiments: T. Kinoshita et al., Science **305**, 1125 (2004) (low density),
  B. Paredes et al., Nature **429**, 277 (2004) (large effective mass)





TG gas theory: FB mapping

- Atoms in tight waveguide $\hbar\omega_\perp \gg \hbar\omega_\ell$, $k_B T$, μ

  Interactions : Longitudinal atom-atom scattering in transverse harmonic trap, M.Olshanii, PRL **81**, 938 (1998)

  g_{1D} renormalized by resonance between longitudinal scattering and excited transverse bound state:

$$g_{1D} = 2a_s\hbar\omega_\perp(1 - 1.460\, a_s/a_\perp)^{-1}$$

$a_s = \lim_{k\to 0} \tan\delta_s(k)/k =$ 3D s-wave scattering length, $a_\perp = \sqrt{2\hbar/m\omega_\perp}$

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- Feshbach resonance tune: $g_{1D} \to +\infty$ (TG gas) as $a_s/a_\perp \to .6848$

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- *FALSE* if motion is 1D AND interactions have hard core (TG)

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- True for *arbitrary* interactions $v_{int}(x_j - x_k)$ and *arbitrary* external potentials $v_{ext}(x_j)$ so long as v_{int} has hard core forcing wave functions to vanish when cores overlap: $\psi(x_1, \cdots, x_N) = 0$ if any $|x_j - x_k| < a$

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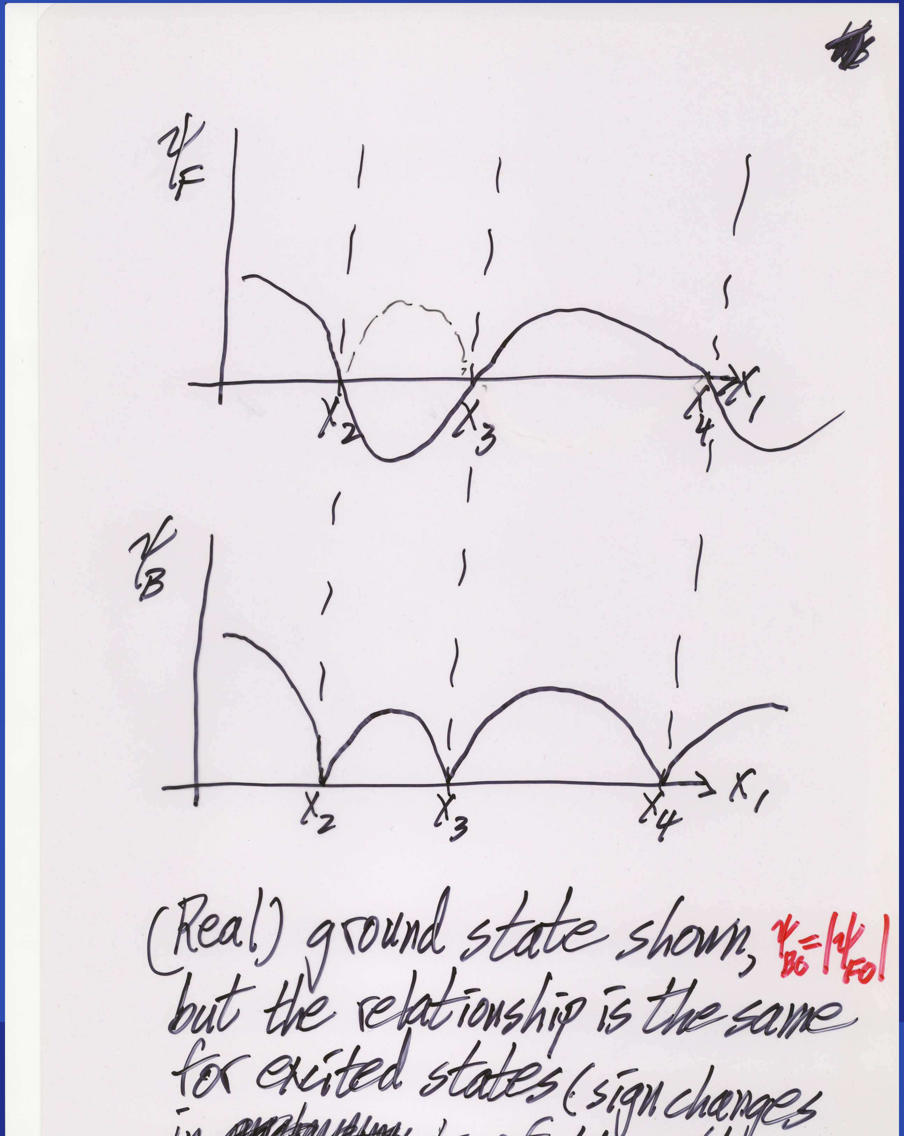
- Generalizes to time-dependent problems: *Theorem:* If v_{ext} is time-dependent and/or initial wave function not an energy eigenstate, single-particle densities $n(x,t)$ generated by time-dependent Schrödinger equation for Bose and Fermi wave functions ψ_B and ψ_F are *equal*:

M.D. Girardeau and E.M. Wright, Phys. Rev. Lett. **84**, 5239 and 5691 (2000)

TG gas theory: FB mapping



Review of FB mapping method to 2005

V.I. Yukalov and M.D. Girardeau, Laser Phys. **2**, 375 (2005)

Bosonic sTG gas

- Theoretical prediction: G.E. Astrakharchik, D. Blume, S. Giorgini, and B.E. Granger, Phys. Rev. Lett. **92**, 030402 (2004), M.T. Batchelor, M. Bortz, X.-W. Guan, and N. Oelkers, J. Stat. Mech. L10001 (2005), G.E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. **95**, 190407 (2005)

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- Recent observation (Innsbruck): E. Haller, M. Gustavsson, M.J. Mark, J.G. Danzl, R. Hart, G. Pupillo, and H.-C. Nägerl, Science **325**, 1224 (2009)

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- Explanation of metastability via FB mapping:
  Exact ground state of trapped TG gas is known for arbitrary number of particles N, via FB mapping to the trapped ideal Fermi gas (next slide):

  M.D. Girardeau, E.M. Wright, and J.M. Triscari, Phys. Rev. A **63**, 033601 (2001)

Bosonic sTG gas

TG gas: Repulsive interactions $g_{1D}^{B}\delta(x_j - x_k)$ with $g_{1D}^{B} \to +\infty$, hence $\gamma_{1D}^{B} \to +\infty$.

Conveniently treated as a constraint on allowed wave functions: $\psi_B = 0 \quad \text{if} \quad x_j = x_k, \quad 1 \leq j < k \leq N$.

Spin aligned ideal Fermi gas eigenstates antisymmetric $\Rightarrow$ constraint automatically satisfied.

Then $\psi_B = A(x_1, \cdots, x_N)\psi_F$ where ψ_F is a Slater determinant of 1D harmonic oscillator states $\varphi_n(x) = \text{const.}e^{-Q^2/2}H_n(Q)$, $Q = x/x_{osc}$, $x_{osc} = \sqrt{\hbar/m\omega}$, H_n are Hermite polynomials, and $A = \prod_{1\leq j<k\leq N} \text{sgn}(x_j - x_k)$ is FB mapping function.

Ground state Slater determinant $\Rightarrow$ van der Monde form $\Rightarrow$

$$\psi_{B0} = |\psi_{F0}| = \text{const.} \left[\prod_{1\leq j<k\leq N} |x_{jk}|\right] e^{-\sum_{i=1}^{N} x_i^2 m\omega/\hbar}.$$

Bosonic sTG gas

sTG gas: Suddenly change interaction from strongly repulsive to strongly attractive as in Innsbruck experiment: $\gamma_{1D}^{B} \gg 1 \to \gamma_{1D}^{B} \ll -1$. In TG limit this means $\gamma_{1D}^{B} = +\infty \to \gamma_{1D}^{B} = -\infty \Rightarrow$ $\psi_B =$ const. $\left[\prod_{1 \le j < k \le N} |x_{jk}|\right] e^{-\sum_{i=1}^{N} x_i^2 m\omega/\hbar}$ is *still an exact energy eigenstate*, now *highly excited* because the ground state is now the totally collapsed McGuire ground state, whose energy $\to -\infty$ in TG limit:

J.B. McGuire, J. Math. Phys. **5**, 622 (1964)

It is *completely stable against collapse* in spite of infinitely strong attraction, and is the *TG limit of the sTG state*.

Bosonic sTG gas

Why is TG state still a solution for $\gamma_{1D}^{B} = -\infty$?:

Theorem: TG state realized in limit $|\gamma_{1D}^{B}| \to \infty$ for *both* $\gamma_{1D}^{B} > 0$ and $\gamma_{1D}^{B} < 0$, and even in dissipative case where γ_{1D}^{B} is complex: S. Dürr et al., Phys. Rev. A **79**, 023614 (2009), Eq. (25) ff

Proof: Lieb-Liniger interaction $g_{1D}^{B}\delta(x_j - x_k) \Rightarrow 2[\partial\psi/\partial x_{jk}]_{x_{jk}=0+} = (mg_{1D}^{B}/\hbar^2)\psi(0)$ with cusp (derivative sign change) at origin: E.H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963)

Then Taylor series $\Rightarrow \psi = (2\hbar^2/mg_{1D}^{B}) + |x_{jk}| + \cdots$ if ψ normalized so $[\partial\psi/\partial x_{jk}]_{x_{jk}=0+} = 1$. Then $|g_{1D}^{B}| \to \infty \Rightarrow \psi(0) \to 0 \Rightarrow$ FB mapping to ideal Fermi gas valid $\Rightarrow$ TG.

Proof fails for (bound) ground state with $g_{1D}^{B} \to -\infty \Rightarrow \psi'(0) = \infty$, but TG and sTG states are excited and expandable about $x_{jk} = 0$.

Bosonic sTG gas

$|\gamma_{1D}^{B}|$ large but finite:

M.D. Girardeau and G.E. Astrakharchik, Phys. Rev. A **81**, 061601(R) (2010):

Exact sTG wave function unknown, but ansatz exact in both limits $\gamma_{1D}^{B} \to \pm\infty$ (trapped TG gas) and $\gamma_{1D}^{B} \to 0$ (trapped ideal Bose gas) is

$$\psi_{B\nu}(x_1, \cdots, x_N) = \left[\prod_{1\le j<\ell\le N} f_\nu(|q_{j\ell}|)\right] \prod_{j=1}^{N} \exp\left(-\frac{x_j^2}{2x_{\text{osc}}^2}\right)$$

where $f_\nu(|q_{j\ell}|) = D_\nu(|q_{j\ell}|)e^{q_{j\ell}^2/4}$ and $q_{j\ell} = (x_j - x_\ell)/x_{\text{osc}}$,

$D_\nu =$ parabolic cylinder function $=$ exact $N = 2$ solution.

ν determined by a transcendental equation and $\to 1$ as $\gamma_{1D}^{B} \to -\infty$: S. Franke-Arnold et al., Eur. Phys. J. D. **22**, 373 (2003).

$\psi_{B\nu}$ metastable for $\gamma_{1D}^{B} \ll -1$, highly excited relative to collapsed ground state.

Bosonic sTG gas

Ground state is analog, for trapped system, of McGuire's collapsed cluster state for untrapped system. Also expressible in terms of a D_ν, but with energy $\to -\infty$ as $g_B \to -\infty$ ($a_{\text{1D}} \to 0+$): see Fig. 5 of S. Franke-Arnold et al., Eur. Phys. J. D. **22**, 373 (2003).

For $\gamma_{1D}^B \to -\infty$ ($a_{\text{1D}} \to 0+$) $N = 2$ ground state well approximated by $\psi_{B0} \approx \exp(-|x_1 - x_2|/a_{\text{1D}})e^{-(x_1^2+x_2^2)m\omega/\hbar}$. Satisfies $x_{12} \to 0$ contact condition exactly and becomes exact for all (x_1, x_2) in both limits $a_{\text{1D}} \to 0+$ (total collapse) and $a_{\text{1D}} \to +\infty$ (ideal Bose gas). Generalization to $N > 2$:

$$\psi_{B0} \approx \prod_{j=1}^N \exp\left(-\frac{x_j^2}{2x_{\text{osc}}^2}\right) \prod_{1 \le j < \ell \le N} \exp\left(-\frac{|x_j - x_l|}{a_{\text{1D}}}\right)$$

Bosonic sTG gas

Connection with trapped hard sphere Bose gas:

$N = 2$ sTG wave function for finite negative γ_{1D}^{B} as function of $x = x_1 - x_2$ has a single node at $|x| = x_{\text{node}} = a_{\text{1D}}[1 - (\frac{\nu}{2} + \frac{1}{4})(\frac{a_{\text{1D}}}{x_{\text{osc}}})^2 + \cdots]$, which is very close to $|x| = a_{\text{1D}}$ when $\gamma_{1D}^{B} \ll -1$ ($0 < a_{\text{1D}}/x_{\text{osc}} \ll 1$).

Theorem: When $|x| \geq x_{\text{node}}$, this sTG wave function identical up to normalization with ground state of hard spheres of diameter x_{node}.

Proof: Both wave functions satisfy same Schrödinger equation in this region and vanish at $x = x_{\text{node}}$ and ∞. Then Sturm-Liouville theory $\Rightarrow$ solution for $|x| \geq x_{\text{node}}$ unique up to normalization, $=$ hard sphere ground state, and energies are equal. Should generalize to $N > 2$.

Two sTG states of spinor fermions

- Model: Fermionic atoms in 1D trap in two different hyperfine states labelled as $\uparrow$ and $\downarrow$, kinetic energy plus spin-independent harmonic trap potential plus spin-independent Lieb-Liniger (LL) delta interaction:
$$\hat{H}_{\mathsf{F}} = \sum_{j=1}^{N} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2} x_j^2 \right) + g_{\mathsf{F}} \sum_{1 \le j < \ell \le N} \delta(x_j - x_\ell)$$

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- $\uparrow$ atoms distinguishable from $\downarrow$ atoms $\Rightarrow$ both 3D s-wave and p-wave scattering allowed. To generate sTG states one needs strong s-wave scattering due to a 3D s-wave Feshbach resonance $\Rightarrow$ 1D even-wave resonance and LL delta interaction with large g_{F}.

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- Space-spin antisymmetry $\Rightarrow$ strong $\uparrow\downarrow$ interaction (space-even), no $\uparrow\uparrow$ or $\downarrow\downarrow$ interaction (space-odd).

Two sTG states of spinor fermions

Exact solution for $N = 2$:

- $\gamma_{\mathsf{F}} = 2mg_{\mathsf{F}}/\hbar^2 n \to +\infty$: Relative wave function $\phi(x) = (|x|/x_{\mathsf{osc}})e^{-(|x|/x_{\mathsf{osc}})^2/2} \Rightarrow$ TG state.
  $\gamma_{\mathsf{F}} \to -\infty$: Still an exact energy eigenstate $\Rightarrow$ *completely stable* against collapse to McGuire's state.

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- $-\infty < \gamma_{\mathrm{F}} \ll -1$: One solution same as for Bose sTG gas, $\phi(x) =$ const.$D_\nu(|x|/x_{\mathrm{osc}})$, node $|x| = x_{\mathrm{node}} > 0$, *metastable* against collapse (sTG).

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- Another solution: bound fermion dimer: S. Chen et al., arXiv:1005.0461v2 and Phys. Rev. A **81**, 031608(R) (2010) (untrapped, on a ring).
  $\gamma_\mathsf{F} = -\infty$: TG still exact eigenstate $\Rightarrow$ Transition probability to dimer under $\gamma_\mathsf{F} = +\infty \to -\infty$ *exactly zero*, still $\ll 1$ under $\gamma_\mathsf{F} \gg 1 \to \gamma_\mathsf{F} \ll -1$.

Two sTG states of spinor fermions

Generalization to $N > 2$: M.D. Girardeau, arXiv:1004.2925:

- If $\gamma_{\text{F}} = +\infty$ exact ground state must vanish with cusp at $x_j = x_\ell$ when $\sigma_j =\uparrow$, $\sigma_\ell =\downarrow$ or vice versa, and if $\sigma_j = \sigma_\ell$ it vanishes there by antisymmetry. Explicit form: $\psi_F = M(x_1, \sigma_1; \cdots; x_N, \sigma_N)\psi_{\text{ideal}}$ where $\psi_{\text{ideal}} =$ spinless ideal Fermi ground state, $M = \prod_{1 \le j < \ell \le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)$, $\alpha(x_j, \sigma_j; x_\ell, \sigma_\ell) = (\delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \downarrow} - \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \uparrow})\text{sgn}(x_j - x_\ell) + \delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \uparrow} + \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \downarrow}$. Space-spin symmetric, ± 1, gives singlet TG cusps.

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- Then TG-ideal Fermi gas hybrid ground state $\psi_{F0} = C_N[\prod_{i=1}^{N} e^{-Q_i^2/2} \prod_{1 \le j < \ell \le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)(x_j - x_\ell)$

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- Still an exact energy eigenstate (now highly excited) after switch $\gamma_{\text{F}} = +\infty \to -\infty \Rightarrow$ *absolutely stable*.

Two sTG states of spinor fermions

Fermionic sTG gas:

- Change γ_{F} from $\gg 1$ to $\ll -1$ as in Innsbruck experiment for bosons. Exact ground state not known for finite γ_{F} in trapped case, but following sTG-ideal Fermi hybrid should be a good approximation:
  $\psi_{sTG} \approx [\prod_{1\le j<\ell\le N} \beta(x_j,\sigma_j;x_\ell,\sigma_\ell)] \prod_{j=1}^{N} \exp\left(-\frac{x_j^2}{2x_{\mathrm{osc}}^2}\right)$
  where $\beta = (\delta_{\sigma_j\uparrow}\delta_{\sigma_\ell\uparrow} + \delta_{\sigma_j\downarrow}\delta_{\sigma_\ell\downarrow})x_{j\ell} + (\delta_{\sigma_j\uparrow}\delta_{\sigma_\ell\downarrow} - \delta_{\sigma_j\downarrow}\delta_{\sigma_\ell\uparrow})D_\nu(|x_{j\ell}/x_{\mathrm{osc}}|)e^{x_{j\ell}^2/4x_{\mathrm{osc}}^2}$
  This ψ_{sTG} satisfies contact conditions exactly.

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Fermionic sTG gas:

- Change γ_{F} from $\gg 1$ to $\ll -1$ as in Innsbruck experiment for bosons. Exact ground state not known for finite γ_{F} in trapped case, but following sTG-ideal Fermi hybrid should be a good approximation:
  $\psi_{sTG} \approx [\prod_{1 \leq j < \ell \leq N} \beta(x_j, \sigma_j; x_\ell, \sigma_\ell)] \prod_{j=1}^{N} \exp\left(-\frac{x_j^2}{2x_{\mathrm{osc}}^2}\right)$
  where $\beta = (\delta_{\sigma_j \uparrow} \delta_{\sigma_\ell \uparrow} + \delta_{\sigma_j \downarrow} \delta_{\sigma_\ell \downarrow}) x_{j\ell} + (\delta_{\sigma_j \uparrow} \delta_{\sigma_\ell \downarrow} - \delta_{\sigma_j \downarrow} \delta_{\sigma_\ell \uparrow}) D_\nu(|x_{j\ell}/x_{\mathrm{osc}}|) e^{x_{j\ell}^2/4x_{\mathrm{osc}}^2}$
  This ψ_{sTG} satisfies contact conditions exactly.
- Reduces to TG-ideal Fermi ground state at $\gamma_{\mathrm{F}} = +\infty$, still exact eigenstate at $\gamma_{\mathrm{F}} = -\infty \Rightarrow$ Probability of transition to sTG gas of bound dimers under switch $\gamma_{\mathrm{F}} \gg 1 \rightarrow \ll -1$ is $\ll 1$, contrary to Chen et al.