Duality Mapping Methods for Strongly Correlated 1D Quantum Gases

Marvin D. Girardeau
School of Optical Sciences, University of Arizona, Tucson, AZ, USA

$$: U.S. Army Research Office

Recent collaborators:

Gregory Astrakharchik, Universitat Politecnica de Catalunya, Barcelona, Spain
Anna Minguzzi, Universite' Joseph Fourier and C.N.R.S., Grenoble, France
Ewan Wright, University of Arizona, Tucson, AZ, USA
Adolfo del Campo and Gonzalo Muga, Universidad del Pais Vasco, Bilbao, Spain

This talk is available online on my web page
www.optics.arizona.edu/girardeau

Plan

- FB mapping method for TG and super TG (sTG) gases

  History, experimental realization, FB mapping, sTG gas

- FB mapping method for the fermionic TG (FTG) gas

  FTG definition, FB mapping, pairing, superconductivity

- FB mapping for arbitrary interaction strength
- Multicomponent systems via multicomponent mappings

  TG and FTG mixtures, multicomponent Calogero-Sutherland gas

- Spinor Fermi and Bose gases via spinor mappings

  Spin-1/2 Fermi, Spin-1 Bose, TWO sTG phases: sTG dimer gas vs. sTG-ideal Fermi hybrid

TG gas history

- First treatment of 1D hard-sphere gas: L. Tonks, Phys. Rev. **50**, 955 (1936): *Classical, high temperature, inapplicable to QM of ultracold atomic vapors*

TG gas history

- First treatment of 1D hard-sphere gas: L. Tonks, Phys. Rev. **50**, 955 (1936): *Classical, high temperature, inapplicable to QM of ultracold atomic vapors*
- Exact QM ground state energy: A. Bijl, Physica **4**, 329 (1937), Note II *No derivation*

TG gas history

- First treatment of 1D hard-sphere gas: L. Tonks, Phys. Rev. **50**, 955 (1936): *Classical, high temperature, inapplicable to QM of ultracold atomic vapors*
- Exact QM ground state energy: A. Bijl, Physica **4**, 329 (1937), Note II *No derivation*
- First published derivation:

  T. Nagamiya, Proc. Phys. Math. Soc. Japan **22**, 705 (1940)

TG gas history

- First treatment of 1D hard-sphere gas: L. Tonks, Phys. Rev. **50**, 955 (1936): *Classical, high temperature, inapplicable to QM of ultracold atomic vapors*
- Exact QM ground state energy: A. Bijl, Physica **4**, 329 (1937), Note II *No derivation*
- First published derivation:

  T. Nagamiya, Proc. Phys. Math. Soc. Japan **22**, 705 (1940)
- Later independent derivations:
  H. Stachowiak, Acta Univ. Wratislaviensis **12**, 93 (1960)

  M. Girardeau, J. Math. Phys. **1**, 516 (1960) *Fermi-Bose mapping: Both ground and excited states, many generalizations*

Strongly interacting 1D atomic gases

- Bosons : the Tonks-Girardeau (TG) gas $\equiv$ 1D Bose gas with repulsive interactions $g^B_{1D}\delta(x_1 - x_2)$ with $\gamma^B_{1D} \longrightarrow \infty$
  Remark: $\gamma^B_{1D} = 2mg^B_{1D}/\hbar^2 n \Rightarrow$ the TG limit is reached at *low density* or *large effective mass*

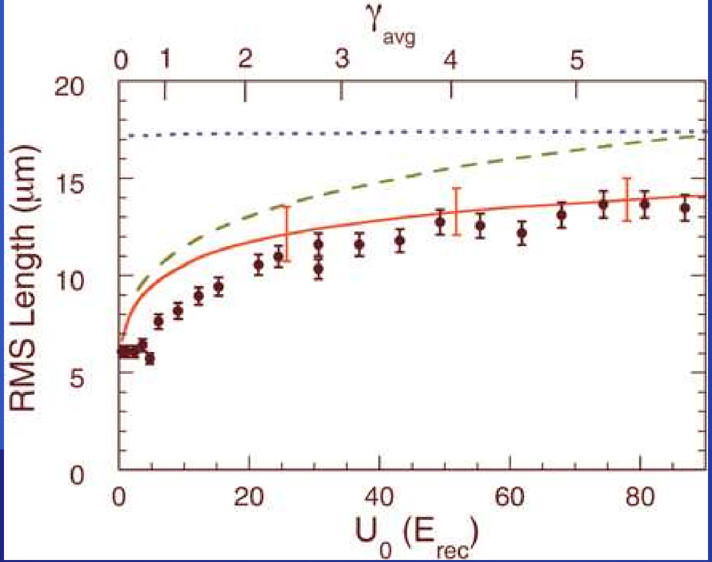
Strongly interacting 1D atomic gases

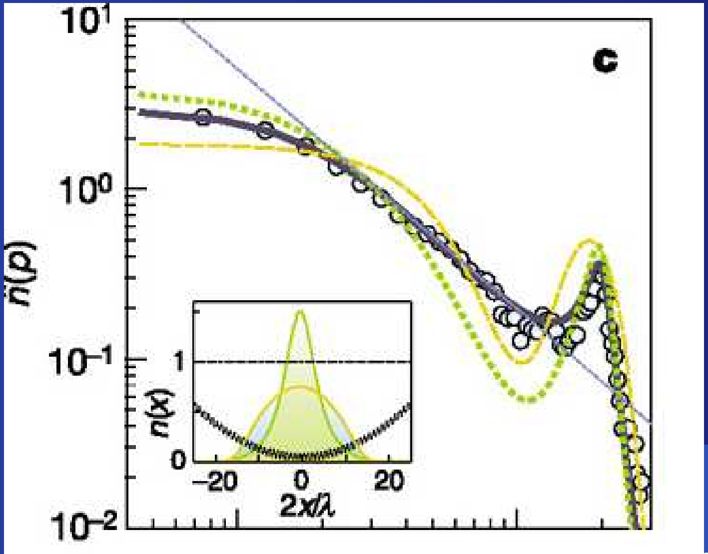
- Bosons : the Tonks-Girardeau (TG) gas $\equiv$ 1D Bose gas with repulsive interactions $g^B_{1D}\delta(x_1 - x_2)$ with $\gamma^B_{1D} \longrightarrow \infty$

  Remark: $\gamma^B_{1D} = 2mg^B_{1D}/\hbar^2 n \Rightarrow$ the TG limit is reached at *low density* or *large effective mass*

- Experiments: T. Kinoshita et al., Science **305**, 1125 (2004) (low density),

  B. Paredes et al., Nature **429**, 277 (2004) (large effective mass)





TG gas theory: FB mapping

- Atoms in tight waveguide $\hbar\omega_\perp \gg \hbar\omega_\ell$, $k_B T$, μ

  Interactions : Longitudinal atom-atom scattering in transverse harmonic trap, M.Olshanii, PRL **81**, 938 (1998)

  g_{1D} renormalized by resonance between longitudinal scattering and excited transverse bound state:

  $$g_{1D} = 2a_s\hbar\omega_\perp(1 - 1.460\, a_s/a_\perp)^{-1}$$

  $a_s = \lim_{k\to 0} \tan\delta_s(k)/k =$ 3D s-wave scattering length, $a_\perp = \sqrt{2\hbar/m\omega_\perp}$

TG gas theory: FB mapping

- Atoms in tight waveguide $\hbar\omega_\perp \gg \hbar\omega_\ell$, $k_B T$, μ

  Interactions : Longitudinal atom-atom scattering in transverse harmonic trap, M.Olshanii, PRL **81**, 938 (1998)

  g_{1D} renormalized by resonance between longitudinal scattering and excited transverse bound state:

  $$g_{1D} = 2a_s\hbar\omega_\perp(1 - 1.460\,a_s/a_\perp)^{-1}$$

  $a_s = \lim_{k\to 0}\tan\delta_s(k)/k =$ 3D s-wave scattering length, $a_\perp = \sqrt{2\hbar/m\omega_\perp}$

- Feshbach resonance tune: $g_{1D} \to +\infty$ (TG gas) as $a_s/a_\perp \to .6848$

TG gas theory: FB mapping

- Atoms in tight waveguide $\hbar\omega_\perp \gg \hbar\omega_\ell$, $k_B T$, μ

  Interactions : Longitudinal atom-atom scattering in transverse harmonic trap, M.Olshanii, PRL **81**, 938 (1998)

  g_{1D} renormalized by resonance between longitudinal scattering and excited transverse bound state:

  $$g_{1D} = 2a_s\hbar\omega_\perp(1 - 1.460\, a_s/a_\perp)^{-1}$$

  $a_s = \lim_{k\to 0} \tan\delta_s(k)/k =$ 3D s-wave scattering length, $a_\perp = \sqrt{2\hbar/m\omega_\perp}$

- Feshbach resonance tune: $g_{1D} \to +\infty$ (TG gas) as $a_s/a_\perp \to .6848$

- *Folk theorem:* Fermi ground state energy $>$ Bose ground state energy due to exclusion principle

TG gas theory: FB mapping

- Atoms in tight waveguide $\hbar\omega_\perp \gg \hbar\omega_\ell$, $k_B T$, μ

  Interactions : Longitudinal atom-atom scattering in transverse harmonic trap, M.Olshanii, PRL **81**, 938 (1998)

  g_{1D} renormalized by resonance between longitudinal scattering and excited transverse bound state:

  $$g_{1D} = 2a_s\hbar\omega_\perp(1 - 1.460\, a_s/a_\perp)^{-1}$$

  $a_s = \lim_{k\to 0} \tan\delta_s(k)/k =$ 3D s-wave scattering length, $a_\perp = \sqrt{2\hbar/m\omega_\perp}$

- Feshbach resonance tune: $g_{1D} \to +\infty$ (TG gas) as $a_s/a_\perp \to .6848$
- *Folk theorem:* Fermi ground state energy $>$ Bose ground state energy due to exclusion principle
- *FALSE* if motion is 1D AND interactions have hard core (TG)

TG gas theory: FB mapping

- *True theorem:* For 1D hard-core particles *complete Bose and Fermi energy spectra* are *identical* (Fermi-Bose duality): M. Girardeau, J. Math. Phys. **1**, 516 (1960)

TG gas theory: FB mapping

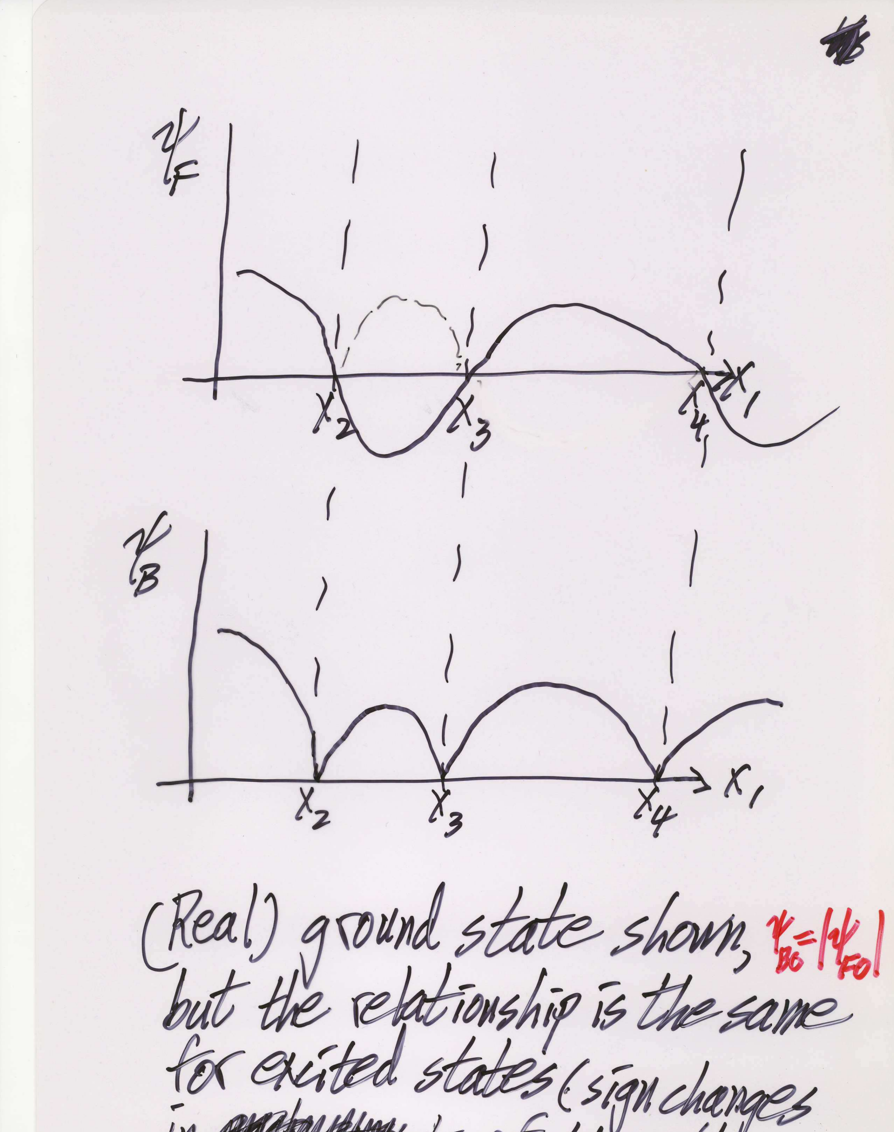
- *True theorem:* For 1D hard-core particles *complete Bose and Fermi energy spectra* are *identical* (Fermi-Bose duality): M. Girardeau, J. Math. Phys. **1**, 516 (1960)
- True for *arbitrary* interactions $v_{int}(x_j - x_k)$ and *arbitrary* external potentials $v_{ext}(x_j)$ so long as v_{int} has hard core forcing wave functions to vanish when cores overlap: $\psi(x_1, \cdots, x_N) = 0$ if any $|x_j - x_k| < a$

TG gas theory: FB mapping

- *True theorem:* For 1D hard-core particles *complete Bose and Fermi energy spectra* are *identical* (Fermi-Bose duality): M. Girardeau, J. Math. Phys. **1**, 516 (1960)
- True for *arbitrary* interactions $v_{int}(x_j - x_k)$ and *arbitrary* external potentials $v_{ext}(x_j)$ so long as v_{int} has hard core forcing wave functions to vanish when cores overlap: $\psi(x_1, \cdots, x_N) = 0$ if any $|x_j - x_k| < a$
- Generalizes to time-dependent problems: *Theorem:* If v_{ext} is time-dependent and/or initial wave function not an energy eigenstate, single-particle densities $n(x,t)$ generated by time-dependent Schrödinger equation for Bose and Fermi wave functions ψ_B and ψ_F are *equal*:

M.D. Girardeau and E.M. Wright, Phys. Rev. Lett. **84**, 5239 and 5691 (2000)

TG gas theory: FB mapping

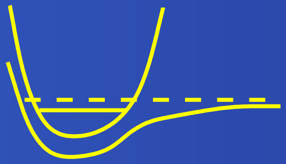


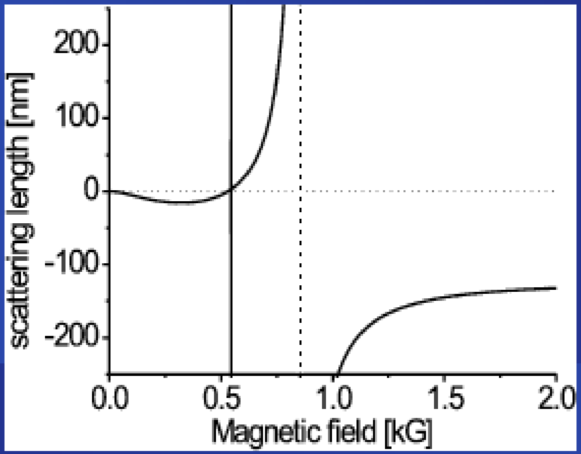
(Real) ground state shown, $\psi_{BG} = |\psi_{FG}|$
but the relationship is the same
for excited states (sign changes

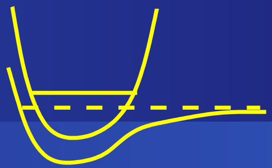
Feshbach resonances

- The scattering length can be tuned by applying an external magnetic field.
  The resonance in the effective 1D coupling constant g_{1D} is due to approach of a bound state to the continuum, and this bound state is in virtually excited transverse modes:

  T. Bergeman, M. Moore, and M. Olshanii, Phys. Rev. Lett. **91**, 163201 (2003).







Bosonic sTG gas

- Theoretical prediction: G.E. Astrakharchik, D. Blume, S. Giorgini, and B.E. Granger, Phys. Rev. Lett. **92**, 030402 (2004), M.T. Batchelor, M. Bortz, X.-W. Guan, and N. Oelkers, J. Stat. Mech. L10001 (2005), G.E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. **95**, 190407 (2005)

Bosonic sTG gas

- Theoretical prediction: G.E. Astrakharchik, D. Blume, S. Giorgini, and B.E. Granger, Phys. Rev. Lett. **92**, 030402 (2004), M.T. Batchelor, M. Bortz, X.-W. Guan, and N. Oelkers, J. Stat. Mech. L10001 (2005), G.E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. **95**, 190407 (2005)
- Recent observation (Innsbruck): E. Haller, M. Gustavsson, M.J. Mark, J.G. Danzl, R. Hart, G. Pupillo, and H.-C. Nägerl, Science **325**, 1224 (2009)

Bosonic sTG gas

- Theoretical prediction: G.E. Astrakharchik, D. Blume, S. Giorgini, and B.E. Granger, Phys. Rev. Lett. **92**, 030402 (2004), M.T. Batchelor, M. Bortz, X.-W. Guan, and N. Oelkers, J. Stat. Mech. L10001 (2005), G.E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. **95**, 190407 (2005)
- Recent observation (Innsbruck): E. Haller, M. Gustavsson, M.J. Mark, J.G. Danzl, R. Hart, G. Pupillo, and H.-C. Nägerl, Science **325**, 1224 (2009)
- Explanation of metastability via FB mapping:
  Exact ground state of trapped TG gas is known for arbitrary number of particles N, via FB mapping to the trapped ideal Fermi gas (next slide):
  M.D. Girardeau, E.M. Wright, and J.M. Triscari, Phys. Rev. A **63**, 033601 (2001)

Bosonic sTG gas

TG gas: Repulsive interactions $g_{1D}^{B}\delta(x_j - x_k)$ with $g_{1D}^{B} \to +\infty$, hence $\gamma_{1D}^{B} \to +\infty$.

Conveniently treated as a constraint on allowed wave functions: $\psi_B = 0 \quad$ if $\quad x_j = x_k, \quad 1 \leq j < k \leq N$.

Spin aligned ideal Fermi gas eigenstates antisymmetric $\Rightarrow$ constraint automatically satisfied.

Then $\psi_B = A(x_1, \cdots, x_N)\psi_F$ where ψ_F is a Slater determinant of 1D harmonic oscillator states $\varphi_n(x) =$ const.$e^{-Q^2/2}H_n(Q)$, $Q = x/x_{osc}$, $x_{osc} = \sqrt{\hbar/m\omega}$, H_n are Hermite polynomials, and $A = \prod_{1\leq j<k\leq N}$ sgn$(x_j - x_k)$ is FB mapping function.

Ground state Slater determinant $\Rightarrow$ van der Monde form $\Rightarrow$ $\psi_{B0} = |\psi_{F0}| =$ const. $\left[\prod_{1\leq j<k\leq N} |x_{jk}|\right] e^{-\sum_{i=1}^{N} x_i^2 m\omega/\hbar}$.

Bosonic sTG gas

sTG gas: Suddenly change interaction from strongly repulsive to strongly attractive as in Innsbruck experiment: $\gamma_{1D}^{B} \gg 1 \rightarrow \gamma_{1D}^{B} \ll -1$. In TG limit this means $\gamma_{1D}^{B} = +\infty \rightarrow \gamma_{1D}^{B} = -\infty \Rightarrow$ $\psi_B =$ const. $\left[\prod_{1 \le j < k \le N} |x_{jk}|\right] e^{-\sum_{i=1}^{N} x_i^2 m\omega/\hbar}$ is *still an exact energy eigenstate*, now *highly excited* because the ground state is now the totally collapsed McGuire ground state, whose energy $\rightarrow -\infty$ in TG limit:

J.B. McGuire, J. Math. Phys. **5**, 622 (1964)

It is *completely stable against collapse* in spite of infinitely strong attraction, and is the *TG limit of the sTG state*.

Bosonic sTG gas

Why is TG state still a solution for $\gamma^B_{1D} = -\infty$?:

Theorem: TG state realized in limit $|\gamma^B_{1D}| \to \infty$ for *both* $\gamma^B_{1D} > 0$ and $\gamma^B_{1D} < 0$, and even in dissipative case where γ^B_{1D} is complex: S. Dürr et al., Phys. Rev. A **79**, 023614 (2009), Eq. (25) ff

Proof: Lieb-Liniger interaction $g^B_{1D}\delta(x_j - x_k) \Rightarrow$ $2[\partial\psi/\partial x_{jk}]_{x_{jk}=0+} = (mg^B_{1D}/\hbar^2)\psi(0)$ with cusp (derivative sign change) at origin: E.H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963) Then Taylor series $\Rightarrow \psi = (2\hbar^2/mg^B_{1D}) + |x_{jk}| + \cdots$ if ψ normalized so $[\partial\psi/\partial x_{jk}]_{x_{jk}=0+} = 1$. Then $|g^B_{1D}| \to \infty \Rightarrow$ $\psi(0) \to 0 \Rightarrow$ FB mapping to ideal Fermi gas valid $\Rightarrow$ TG. Proof fails for (bound) ground state with $g^B_{1D} \to -\infty \Rightarrow$ $\psi'(0) = \infty$, but TG and sTG states are excited and expandable about $x_{jk} = 0$.

Bosonic sTG gas

$|\gamma_{1D}^{B}|$ large but finite:

M.D. Girardeau and G.E. Astrakharchik, Phys. Rev. A **81**, 061601(R) (2010):

Exact sTG wave function unknown, but ansatz exact in both limits $\gamma_{1D}^{B} \to \pm\infty$ (trapped TG gas) and $\gamma_{1D}^{B} \to 0$ (trapped ideal Bose gas) is

$$\psi_{B\nu}(x_1, \cdots, x_N) = \left[\prod_{1\le j<\ell\le N} f_\nu(|q_{j\ell}|)\right] \prod_{j=1}^{N} \exp\left(-\frac{x_j^2}{2x_{\text{osc}}^2}\right)$$

where $f_\nu(|q_{j\ell}|) = D_\nu(|q_{j\ell}|)e^{q_{j\ell}^2/4}$ and $q_{j\ell} = (x_j - x_\ell)/x_{\text{osc}}$,

$D_\nu =$ parabolic cylinder function $=$ exact $N = 2$ solution.

ν determined by a transcendental equation and $\to 1$ as $\gamma_{1D}^{B} \to -\infty$: S. Franke-Arnold et al., Eur. Phys. J. D. **22**, 373 (2003).

$\psi_{B\nu}$ metastable for $\gamma_{1D}^{B} \ll -1$, highly excited relative to collapsed ground state.

Bosonic sTG gas

Ground state is analog, for trapped system, of McGuire's collapsed cluster state for untrapped system. Also expressible in terms of a D_ν, but with energy $\to -\infty$ as $g_B \to -\infty$ ($a_{\text{1D}} \to 0+$): see Fig. 5 of S. Franke-Arnold et al., Eur. Phys. J. D. **22**, 373 (2003).

For $\gamma_{1D}^B \to -\infty$ ($a_{\text{1D}} \to 0+$) $N = 2$ ground state well approximated by $\psi_{B0} \approx \exp(-|x_1 - x_2|/a_{\text{1D}})e^{-(x_1^2+x_2^2)m\omega/\hbar}$. Satisfies $x_{12} \to 0$ contact condition exactly and becomes exact for all (x_1, x_2) in both limits $a_{\text{1D}} \to 0+$ (total collapse) and $a_{\text{1D}} \to +\infty$ (ideal Bose gas). Generalization to $N > 2$:

$$\psi_{B0} \approx \prod_{j=1}^N \exp\left(-\frac{x_j^2}{2x_{\text{osc}}^2}\right) \prod_{1\le j<\ell\le N} \exp\left(-\frac{|x_j - x_l|}{a_{\text{1D}}}\right)$$

Bosonic sTG gas

Connection with trapped hard sphere Bose gas:

$N=2$ sTG wave function for finite negative γ_{1D}^{B} as function of $x=x_1-x_2$ has a single node at $|x|=x_{\text{node}}=a_{\text{1D}}[1-(\frac{\nu}{2}+\frac{1}{4})(\frac{a_{\text{1D}}}{x_{\text{osc}}})^2+\cdots]$, which is very close to $|x|=a_{\text{1D}}$ when $\gamma_{1D}^{B}\ll -1$ ($0<a_{\text{1D}}/x_{\text{osc}}\ll 1$).

Theorem: When $|x|\geq x_{\text{node}}$, this sTG wave function identical up to normalization with ground state of hard spheres of diameter x_{node}.

Proof: Both wave functions satisfy same Schrödinger equation in this region and vanish at $x=x_{\text{node}}$ and ∞. Then Sturm-Liouville theory $\Rightarrow$ solution for $|x|\geq x_{\text{node}}$ unique up to normalization, $=$ hard sphere ground state, and energies are equal. Should generalize to $N>2$.

Fermionic TG (FTG) gas, FB mapping

$$\mathcal{H} = \sum_i -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j} v(x_i - x_j)$$

- Fermions : antisymmetric spatial wavefunction
  $\Rightarrow$ no s-wave but p-wave (“odd-wave” in 1D)

Fermionic TG (FTG) gas, FB mapping

$$\mathcal{H} = \sum_i -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j} v(x_i - x_j)$$

- Fermions : antisymmetric spatial wavefunction
  $\Rightarrow$ no s-wave but p-wave (“odd-wave” in 1D)
- p-wave interactions as boundary conditions...

$$\frac{1}{\psi_F(x)}\frac{d\psi_F(x)}{dx}\bigg|_{x=0+} = -\frac{2\hbar^2}{m g_{1D}^F}$$

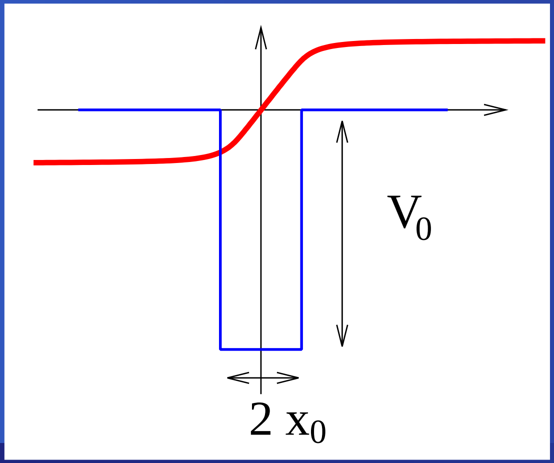
Fermionic TG (FTG) gas, FB mapping

$$\mathcal{H} = \sum_i -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j} v(x_i - x_j)$$

- Fermions : antisymmetric spatial wavefunction
  $\Rightarrow$ no s-wave but p-wave (“odd-wave” in 1D)
- p-wave interactions as boundary conditions...

$$\frac{1}{\psi_F(x)}\frac{d\psi_F(x)}{dx}\bigg|_{x=0+} = -\frac{2\hbar^2}{m g_{1D}^F}$$

- ... or square-well model for the interaction potential



$$\psi_F(x = x_0) = -\psi_F(x = -x_0) = -a_{1D}^F \psi_F'(x = \pm x_0)$$

$$\psi_F(x) = \mathrm{sign}(x) \qquad |x| > x_0$$

$$\psi_F(x) = \sin(\kappa x) \qquad -x_0 < x < x_0$$

$$\kappa = \sqrt{mV_0/\hbar^2} = \pi/2x_0$$

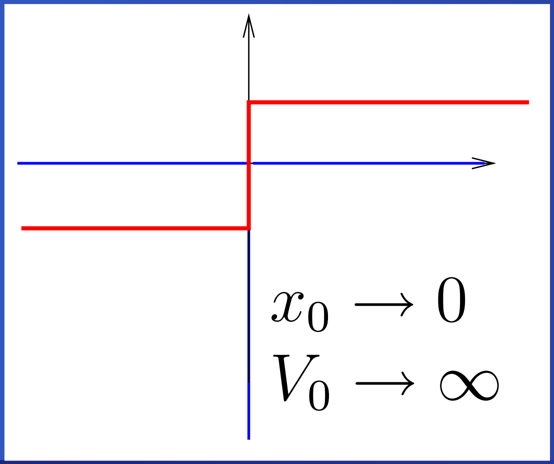
Fermionic TG (FTG) gas, FB mapping

$$\mathcal{H} = \sum_i -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j} v(x_i - x_j)$$

- Fermions : antisymmetric spatial wavefunction
  $\Rightarrow$ no s-wave but p-wave (“odd-wave” in 1D)
- p-wave interactions as boundary conditions...

$$\frac{1}{\psi_F(x)}\frac{d\psi_F(x)}{dx}\bigg|_{x=0+} = -\frac{2\hbar^2}{m g_{1D}^F}$$

- ... or square-well model for the interaction potential



$$\psi_F(x = x_0) = -\psi_F(x = -x_0) = -a_{1D}^F \psi_F'(x = \pm x_0)$$

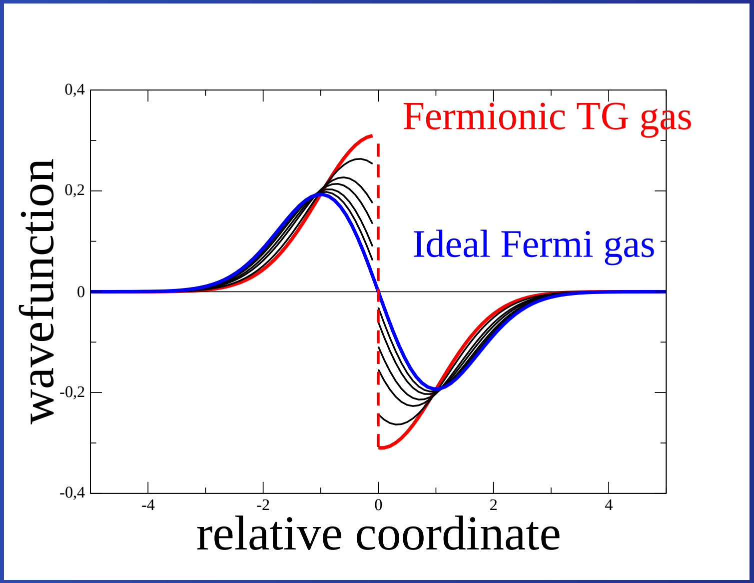
$$\psi_F(x) = \mathrm{sign}(x) \qquad |x| > x_0$$

$$\psi_F(x) = \sin(\kappa x) \qquad -x_0 < x < x_0$$

$$\kappa = \sqrt{mV_0/\hbar^2} = \pi/2x_0$$

From ideal Fermi to FTG gas

- N=2 relative wavefunctions at increasing interaction strength γ^F_{1D} – FTG gas : $\gamma^F_{1D} \to -\infty$



Notice that the absolute value of the FTG wavefunction coincides with the one of an ideal Bose gas

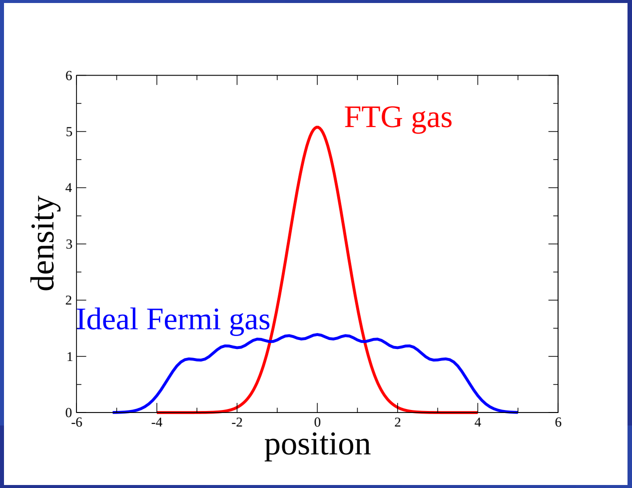
FB mapping for the FTG gas

- Exact N-body ground state [B.E. Granger, D. Blume,PRL **92** 133202 (2004); M. Girardeau and M. Olshanii, cond-mat/0309396]

$$\Psi_{FTG}(x_1, \ldots, x_N) = \mathcal{A}(x_1, \ldots, x_N)\Pi_{j=1}^{N}\phi_0(x_j)$$

  with $\mathcal{A}(x_1, \ldots, x_N) = \Pi_{1\leq j<l\leq N}\mathrm{sign}(x_j - x_l)$ and $\phi_0(x)$ the ground orbital

- Density profile of the fermionic TG gas : the same as for an ideal Bose gas – *much narrower than that of an ideal Fermi gas due to attractive interactions!*



Pairing

- Pairing of fermions with strong attractive interactions we need the analogue of *Penrose and Onsager criterion* for BEC in an interacting system

Pairing

- Pairing of fermions with strong attractive interactions we need the analogue of *Penrose and Onsager criterion* for BEC in an interacting system
- Yang's criterion [C.N. Yang, RMP **34** 694 (1962)]: pairing $\Rightarrow$ ODLRO of the two-body density matrix

$\rho_2(x_1, x_2; x_1', x_2') \equiv N(N-1) \int dx_3..dx_N \Phi^*(x_1, x_2.., x_N)\Phi(x1', x_2', .., x_N)$:

$$\rho_2(x_1, x_2; x_1', x_2') \longrightarrow N\alpha \; u_1(x_1, x_2) u_1(x_1', x_2')$$

for $x_1 \simeq x_2$, $x_1' \simeq x_2'$, and arbitrary pair distance $|x_1 - x_2'|$, with $0 < \alpha < 1$

pair-condensate fraction

Pairing of homogeneous FTG gas

- Analytic expression for the two-body density matrix:

$$\rho_2 = N(N-1)w_1(x_1,x_2)w_1(x_1',x_2')[G(x_1,x_2;x_1',x_2')]^{N-2}$$

$G(x_1,x_2;x_1',x_2') = 1 + 2(y_1 - y_2 + y_3 - y_4)/L$, $w_1(x_1,x_2) = \text{sign}(x_1 - x_2)/L$,

$(y_1 \leq y_2 \leq y_3 \leq y_4)$=sort$(x_1,x_2,x_1',x_2')$

- In the thermodynamic limit :

$$\rho_2 \sim (N/2)u_1(x_1,x_2)u_1(x_1',x_2')$$

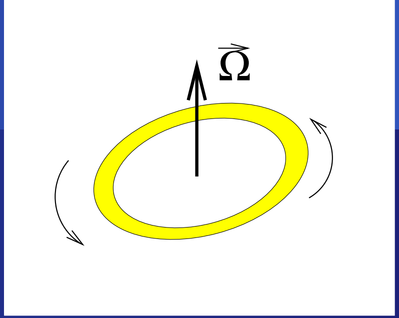
with $u_1(x_1,x_2) = \text{sign}(x_1 - x_2)e^{-2n|x_1-x_2|}$

[M.D. Girardeau and A. Minguzzi, PRL **96**, 080404 (2006)]

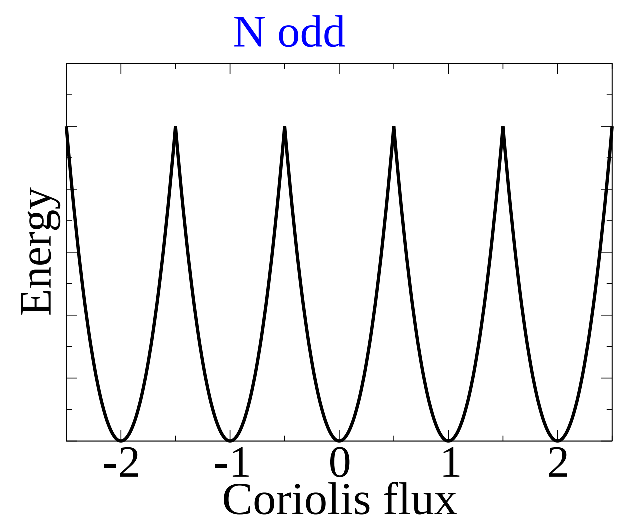
N/2 fermions are paired

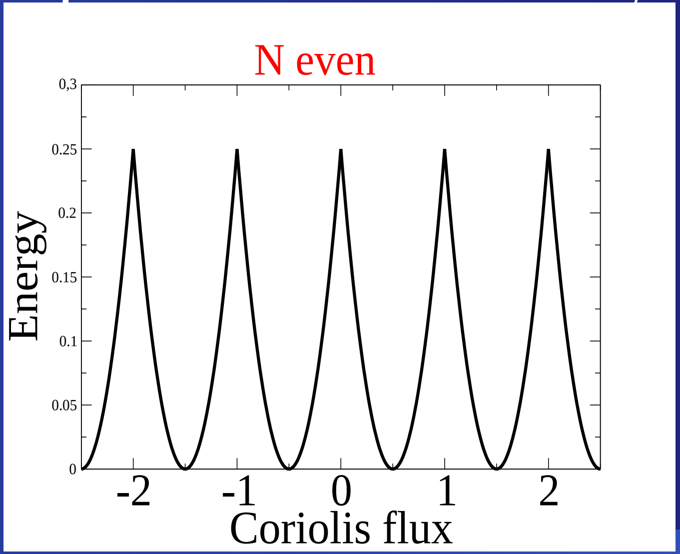
Pairs in between “BEC” and “BCS” regimes

Rotating FTG gas



- Set the ring into rotation :
- Rotation of neutral atoms $\equiv$ magnetic field on charged particles; $H_{kin} = \sum_j \frac{1}{2m}\left[-i\hbar\frac{\partial}{\partial x_j} - \frac{\hbar}{R}\frac{\Phi}{\Phi_0}\right]^2$ with $\Phi = \pi\Omega R^2$, $\Phi_0 = h/2m$ and R ring radius
- Parity effect : energy landscape *vs* Coriolis flux Φ/Φ_0





Arbitrary interaction strength

- Mapping $\psi_B = A(x_1, \cdots, x_N)\psi_F$ and inverse (with $A^{-1} = A$) not restricted to TG or FTG gas; also maps bosons with *repulsive* interactions of *any* coupling constant $g_{1D}^B = 2\hbar^2/m|a_{1D}| > 0$ and spin-aligned fermions with reciprocal *attractive* coupling constant $g_{1D}^F = 2\hbar^2|a_{1D}|/m$ such that 1D scattering lengths are equal: $a_{1D}^B = a_{1D}^F = a_{1D} < 0$:

  T. Cheon and T. Shigehara, Phys. Lett. A **243**, 111 (1998), Phys. Rev. Lett. **82**, 2536 (1999)

  B.E. Granger and D. Blume, Phys. Rev. Lett. **92**, 133202 (2004)

  M.D. Girardeau and M. Olshanii, cond-mat/0309396

  M.D. Girardeau, Hieu Nguyen, and M. Olshanii, Optics Communications **243**, 3 (2004)

Arbitrary interaction strength

- Mapping $\psi_B = A(x_1, \cdots, x_N)\psi_F$ and inverse (with $A^{-1} = A$) not restricted to TG or FTG gas; also maps bosons with *repulsive* interactions of *any* coupling constant $g_{1D}^B = 2\hbar^2/m|a_{1D}| > 0$ and spin-aligned fermions with reciprocal *attractive* coupling constant $g_{1D}^F = 2\hbar^2|a_{1D}|/m$ such that 1D scattering lengths are equal: $a_{1D}^B = a_{1D}^F = a_{1D} < 0$:

  T. Cheon and T. Shigehara, Phys. Lett. A **243**, 111 (1998), Phys. Rev. Lett. **82**, 2536 (1999)

  B.E. Granger and D. Blume, Phys. Rev. Lett. **92**, 133202 (2004)

  M.D. Girardeau and M. Olshanii, cond-mat/0309396

  M.D. Girardeau, Hieu Nguyen, and M. Olshanii, Optics Communications **243**, 3 (2004)
- Mapping allows strongly interacting 1D Fermi gas to be treated via weakly interacting Bose gas and vice versa.

Multicomponent systems and mappings

Soluble 1D Bose and Fermi gas mixture models:

M.D. Girardeau and A. Minguzzi, Phys. Rev. Lett. **99**, 230402 (2007)

Many soluble models with TG and/or FTG interactions.

Example: N_B bosons with TG boson-boson interactions and N_F noninteracting fermions with TG boson-fermion interactions, equal masses $m_B = m_F = m$.

Realizeable by bosonic and fermionic isotopes of a given alkali element, *e.g.* ^{6}Li - ^{7}Li.

Low B and F densities $\Rightarrow$ TG BB interaction and negligible FF interaction, s-wave BF Feshbach resonance $\Rightarrow$ TG BF interaction.

Multicomponent systems and mappings

Exact energy eigenstates:

First construct model wavefunction $\Psi_M(X_B, X_F)$ which is a Slater determinant of $N = N_B + N_F$ orthonormal orbitals $u_1(x), \cdots, u_N(x)$ occupied by all N bosons and fermions, with all possible permutations of the atoms among the orbitals. It vanishes not only at points $x_{jF} = x_{\ell F}$ required by fermionic antisymmetry, but also at points $x_{jB} = x_{\ell B}$ and $x_{jB} = x_{\ell F}$ required by the BB and BF point hard core constraints. Repair its improper BB symmetry and introduce required TG cusps by a mapping function

$$A = \prod_{1 \le j < \ell \le N_B} \mathrm{sgn}(x_{jB} - x_{\ell B}) \prod_{j=1}^{N_B} \prod_{\ell=1}^{N_F} \mathrm{sgn}(x_{jB} - x_{\ell F})$$

Multicomponent systems and mappings

Exact energy eigenstates, continued:

Then physical wavefunctions are

$\Psi(X_B, X_F) = A(X_B, X_F)\Psi_M(X_B, X_F)$.

Ground state is filled Fermi sea of lowest N orbitals, and excited states are generated by choosing higher orbitals.

For harmonic trapping, ground state is a straightforward generalization of that of the trapped TG gas:

$$\Psi_0 = [\prod_{1 \leq j < \ell \leq N_B} |x_{jB} - x_{\ell B}|][\prod_{j=1}^{N_B}\prod_{\ell=1}^{N_F} |x_{jB} - x_{\ell F}|]$$

$$\times [\prod_{1 \leq j < \ell \leq N_F} (x_{jF} - x_{\ell F})]\phi_0(x_{1B}) \cdots \phi_0(x_{N_B B})$$

$\times \phi_0(x_{1F}) \cdots \phi_0(x_{N_F F})$ where $\phi_0(x) = \exp[-(x/x_{\text{osc}})^2/2]$

Multicomponent systems and mappings

Another soluble mixture model:

M.D. Girardeau, Phys. Rev. Lett. **102**, 245303 (2009)

Two different ideal Bose gases A and B (no AA or BB interaction) with an FTG AB interaction induced by a p-wave AB Feshbach resonance. m_A and m_B can be different, and so can N_A and N_B.

Assuming trapping on a ring, the Hamiltonian consists of kinetic energy operators of components A and B plus AB FTG interaction:

$$\hat{H} = \sum_{i=1}^{N_A} \frac{-\hbar^2}{2m_A} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N_B} \frac{-\hbar^2}{2m_B} \frac{\partial^2}{\partial y_i^2} + \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} \hat{v}_o(x_i - y_j)$$

Multicomponent systems and mappings

Another soluble mixture model, continued:

Odd-wave FTG interaction $\hat{v}_o$ representable as $x_0 \to 0$ and $V_0 \to \infty$ limit of a square well of width $2x_0$ and depth V_0, carried out such that $V_0 x_0^2 = (\pi\hbar)^2/8\mu$ where $\mu =$ effective mass. Then scattering length $a_{AB} = -\infty$ (FTG limit). For square well width $2x_0$ nonzero but very small, unnormalized ground state Ψ_0 is constant (say ± 1) when all $|x_i - y_j| > 2x_0$, except for sign changes as each $x_i - y_j$ varies from $-x_0$ to x_0 in accordance with internal wave function $\pm \sin[\kappa(x_i - y_j)]$ where $\kappa x_0 = \pi/2$.

Multicomponent systems and mappings

Another soluble mixture model, continued:

In limit $x_0 \to 0$, Ψ_0 becomes invisible inside wells and Ψ_0 appears to jump discontinuously between ± 1, but there are hidden nodes at $x_i - y_j = 0$. Ψ_0 maps to a model state $\Psi_{M0} =$ two noninteracting ideal Bose gases totally Bose-Einstein condensed into trivial ground orbital $\phi_0 = 1$. Then when $x_0 = 0+$, $\Psi_0 = \Psi_{M0} M = M$ and $\Psi_{M0} = 1$ where

$$M(x_1, \cdots, x_{N_A}; y_1, \cdots, y_{N_B}) = \prod_{i=1}^{N_A} \prod_{j=1}^{N_B} \mathrm{sgn}(x_i - y_j)$$

Mapped bosonic state Ψ_{M0} is constant ± 1 outside square wells, interior wave function $\pm \sin(\kappa |x_i - y_j|)$ vanishes with cusps at $x_i - y_j = 0$.

Multicomponent systems and mappings

Another soluble mixture model, continued:

Densities of components A and B are trivial constants, but two-particle density matrices of Ψ_0 are nontrivial:

If $x_1 < x_2 < x_1' < x_2'$ one finds $\rho_{2AA} = n_A^2 e^{-2n_B|x_1 - x_2|} e^{-2n_B|x_1' - x_2'|}$. Previous Yang argument for one-component FTG gas $\Rightarrow$ if variable pairs (x_1, x_2) and (x_1', x_2') are separated to arbitrary distance while keeping $|x_1 - x_2|$ and $|x_1' - x_2'|$ fixed, then ρ_{2AA} remains constant, signalling AA-pair ODLRO. Since AA interaction is zero, this is an indirect effect of strong FTG AB attraction.

Nevertheless, there is no AB-pair ODLRO:

If $x' = x + d$ and $y' = y + d$, then $\rho_{2AB} = n_A n_B e^{-2(n_A + n_B)d}$ which vanishes exponentially as $d \to \infty$.

Multicomponent systems and mappings

Generalities of 1D duality mappings:

General rule for constructing a mapping from ideal (noninteracting) gas mixture to strongly interacting mixture:

TG interactions due to an s-wave resonance: Mapping factor sgn$(x-y)$ produces cusp at $x-y=0$ due to even-wave TG (point hard core) interaction between an atom at x and one at y, starting from an ideal gas mixture wave function with a Fermi-like node at $x-y=0$.

FTG interactions due to a p-wave resonance: Mapping factor sgn$(x-y)$ produces wave function sign change at $x-y=0$ due to odd-wave FTG interaction between an atom at x and one at y, starting from an ideal gas mixture wave function which does *not* vanish at $x-y=0$.

Multicomponent systems and mappings

Calogero-Sutherland (CS) gas:

M.D. Girardeau and G.E. Astrakharchik, Phys. Rev. A **81**, 043601 (2010)

Duality mappings discussed thus far generate exact solutions of systems with TG and/or FTG interactions starting from ideal (noninteracting) gases. More generally, one can introduce TG and/or FTG interactions starting from *any* exactly soluble 1D model.

One example is the Bose gas on a ring of circumference L with CS interatomic interactions:

F. Calogero, J. Math. Phys. **10**, 2191, 2197 (1969); B. Sutherland, *12*, 246 (1971)

CS interaction is the L-periodic extension of an inverse square potential, or the interaction corresponding to the chord distance $L\sin(\pi x/L)$ between two such particles:

Multicomponent systems and mappings

CS gas, continued:

$$V^{\mathrm{CS}}(x_{ij}) = \frac{\pi^2\hbar^2}{mL^2}\frac{\lambda(\lambda-1)}{\sin^2(\pi x_{ij}/L)}$$

where $\lambda \geq 0 =$ interaction parameter.

Exact ground state:

$$\psi_0(x_1, ..., x_N) = \prod_{i<j}^{N} \left| \sin \frac{\pi(x_i - x_j)}{L} \right|^{\lambda}$$

Useful limits: $\lambda = 0 \Rightarrow$ ideal Bose gas, $\lambda = 1 \Rightarrow$ TG gas, λ slightly $> 1 \Rightarrow$ long-range behavior of sTG gas, $\lambda > 2 \Rightarrow$ quasicrystal. *CS model interpolates between all these other soluble models*.

Multicomponent systems and mappings

Mixture of two CS gases A and B with FTG AB interaction:

No AB interaction $\Rightarrow$ AB ground state $= \psi_{0A}\psi_{0B}$.

Generate FTG AB interaction by same mapping used for a mixture of two ideal Bose gases with FTG AB interaction:

$$M(x_1, \cdots, x_{N_A}; y_1, \cdots, y_{N_B}) = \prod_{i=1}^{N_A}\prod_{j=1}^{N_B} \mathrm{sgn}(x_i - y_j)$$

Then

$$\psi_0 = M\psi_{0A}\psi_{0B} \quad = \prod_{1\le 1<j\le N_A}\ \prod_{1\le k<\ell\le N_B} |\sin\pi(x_i - x_j)/L|^{\lambda_A}$$
$$\times \quad |\sin\pi(y_k - y_\ell)/L|^{\lambda_B}\,\mathrm{sgn}(x_i - y_k)$$

Properties: Use Monte Carlo. See our paper.

Two sTG states of spinor fermions

- Model: Fermionic atoms in 1D trap in two different hyperfine states labelled as ↑ and ↓, kinetic energy plus spin-independent harmonic trap potential plus spin-independent Lieb-Liniger (LL) delta interaction:

$$\hat{H}_{\mathsf{F}} = \sum_{j=1}^{N} \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2} x_j^2 \right) + g_{\mathsf{F}} \sum_{1 \le j < \ell \le N} \delta(x_j - x_\ell)$$

Two sTG states of spinor fermions

- Model: Fermionic atoms in 1D trap in two different hyperfine states labelled as ↑ and ↓, kinetic energy plus spin-independent harmonic trap potential plus spin-independent Lieb-Liniger (LL) delta interaction:
  $$\hat{H}_{\mathsf{F}} = \sum_{j=1}^{N} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2} x_j^2 \right) + g_{\mathsf{F}} \sum_{1 \le j < \ell \le N} \delta(x_j - x_\ell)$$
- ↑ atoms distinguishable from ↓ atoms ⇒ both 3D s-wave and p-wave scattering allowed. To generate sTG states one needs strong s-wave scattering due to a 3D s-wave Feshbach resonance ⇒ 1D even-wave resonance and LL delta interaction with large g_{F}.

Two sTG states of spinor fermions

- Model: Fermionic atoms in 1D trap in two different hyperfine states labelled as $\uparrow$ and $\downarrow$, kinetic energy plus spin-independent harmonic trap potential plus spin-independent Lieb-Liniger (LL) delta interaction:
$$\hat{H}_{\mathsf{F}} = \sum_{j=1}^{N} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2} x_j^2 \right) + g_{\mathsf{F}} \sum_{1 \le j < \ell \le N} \delta(x_j - x_\ell)$$
- $\uparrow$ atoms distinguishable from $\downarrow$ atoms $\Rightarrow$ both 3D s-wave and p-wave scattering allowed. To generate sTG states one needs strong s-wave scattering due to a 3D s-wave Feshbach resonance $\Rightarrow$ 1D even-wave resonance and LL delta interaction with large g_{F}.
- Space-spin antisymmetry $\Rightarrow$ strong $\uparrow\downarrow$ interaction (space-even), no $\uparrow\uparrow$ or $\downarrow\downarrow$ interaction (space-odd).

Two sTG states of spinor fermions

Exact solution for $N = 2$:

- $\gamma_{\mathsf{F}} = 2mg_{\mathsf{F}}/\hbar^2 n \rightarrow +\infty$: Relative wave function $\phi(x) = (|x|/x_{\mathsf{osc}})e^{-(|x|/x_{\mathsf{osc}})^2/2} \Rightarrow$ TG state.
  $\gamma_{\mathsf{F}} \rightarrow -\infty$: Still an exact energy eigenstate $\Rightarrow$ *completely stable* against collapse to McGuire’s state.

Two sTG states of spinor fermions

Exact solution for $N = 2$:

- $\gamma_{\mathsf{F}} = 2mg_{\mathsf{F}}/\hbar^2 n \to +\infty$: Relative wave function $\phi(x) = (|x|/x_{\mathsf{osc}})e^{-(|x|/x_{\mathsf{osc}})^2/2} \Rightarrow$ TG state. $\gamma_{\mathsf{F}} \to -\infty$: Still an exact energy eigenstate $\Rightarrow$ *completely stable* against collapse to McGuire's state.
- $-\infty < \gamma_{\mathsf{F}} \ll -1$: One solution same as for Bose sTG gas, $\phi(x) =$ const.$D_\nu(|x|/x_{\mathsf{osc}})$, node $|x| = x_{\mathsf{node}} > 0$, *metastable* against collapse (sTG).

Two sTG states of spinor fermions

Exact solution for $N = 2$:

- $\gamma_{\mathsf{F}} = 2mg_{\mathsf{F}}/\hbar^2 n \rightarrow +\infty$: Relative wave function $\phi(x) = (|x|/x_{\mathsf{osc}})e^{-(|x|/x_{\mathsf{osc}})^2/2} \Rightarrow$ TG state. $\gamma_{\mathsf{F}} \rightarrow -\infty$: Still an exact energy eigenstate $\Rightarrow$ *completely stable* against collapse to McGuire's state.
- $-\infty < \gamma_{\mathsf{F}} \ll -1$: One solution same as for Bose sTG gas, $\phi(x) =$ const.$D_\nu(|x|/x_{\mathsf{osc}})$, node $|x| = x_{\mathsf{node}} > 0$, *metastable* against collapse (sTG).
- Another solution: bound fermion dimer: S. Chen et al., arXiv:1005.0461v2 and Phys. Rev. A **81**, 031608(R) (2010) (untrapped, on a ring). $\gamma_{\mathsf{F}} = -\infty$: TG still exact eigenstate $\Rightarrow$ Transition probability to dimer under $\gamma_{\mathsf{F}} = +\infty \rightarrow -\infty$ *exactly zero*, still $\ll 1$ under $\gamma_{\mathsf{F}} \gg 1 \rightarrow \gamma_{\mathsf{F}} \ll -1$.

Two sTG states of spinor fermions

Generalization to $N > 2$: M.D. Girardeau, arXiv:1004.2925:

- If $\gamma_{\text{F}} = +\infty$ exact ground state must vanish with cusp at $x_j = x_\ell$ when $\sigma_j = \uparrow$, $\sigma_\ell = \downarrow$ or vice versa, and if $\sigma_j = \sigma_\ell$ it vanishes there by antisymmetry. Explicit form: $\psi_F = M(x_1, \sigma_1; \cdots ; x_N, \sigma_N)\psi_{\text{ideal}}$ where $\psi_{\text{ideal}} =$ spinless ideal Fermi ground state, $M = \prod_{1 \le j < \ell \le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)$, $\alpha(x_j, \sigma_j; x_\ell, \sigma_\ell) = (\delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \downarrow} - \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \uparrow})\text{sgn}(x_j - x_\ell) + \delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \uparrow} + \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \downarrow}$. Space-spin symmetric, ± 1, gives singlet TG cusps.

Two sTG states of spinor fermions

Generalization to $N > 2$: M.D. Girardeau, arXiv:1004.2925:

- If $\gamma_{\text{F}} = +\infty$ exact ground state must vanish with cusp at $x_j = x_\ell$ when $\sigma_j = \uparrow$, $\sigma_\ell = \downarrow$ or vice versa, and if $\sigma_j = \sigma_\ell$ it vanishes there by antisymmetry. Explicit form: $\psi_F = M(x_1, \sigma_1; \cdots ; x_N, \sigma_N)\psi_{\text{ideal}}$ where $\psi_{\text{ideal}} =$ spinless ideal Fermi ground state, $M = \prod_{1 \le j < \ell \le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)$, $\alpha(x_j, \sigma_j; x_\ell, \sigma_\ell) = (\delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \downarrow} - \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \uparrow})\text{sgn}(x_j - x_\ell) + \delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \uparrow} + \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \downarrow}$. Space-spin symmetric, ± 1, gives singlet TG cusps.
- Then TG-ideal Fermi gas hybrid ground state $\psi_{F0} = C_N [\prod_{i=1}^N e^{-Q_i^2/2} \prod_{1 \le j < \ell \le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)(x_j - x_\ell)$

Two sTG states of spinor fermions

Generalization to $N > 2$: M.D. Girardeau, arXiv:1004.2925:

- If $\gamma_{\mathrm{F}} = +\infty$ exact ground state must vanish with cusp at $x_j = x_\ell$ when $\sigma_j = \uparrow$, $\sigma_\ell = \downarrow$ or vice versa, and if $\sigma_j = \sigma_\ell$ it vanishes there by antisymmetry. Explicit form: $\psi_F = M(x_1, \sigma_1; \cdots; x_N, \sigma_N)\psi_{\mathrm{ideal}}$ where $\psi_{\mathrm{ideal}} =$ spinless ideal Fermi ground state, $M = \prod_{1 \le j < \ell \le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)$, $\alpha(x_j, \sigma_j; x_\ell, \sigma_\ell) = (\delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \downarrow} - \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \uparrow})\mathrm{sgn}(x_j - x_\ell) + \delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \uparrow} + \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \downarrow}$. Space-spin symmetric, ± 1, gives singlet TG cusps.

- Then TG-ideal Fermi gas hybrid ground state $\psi_{F0} = C_N[\prod_{i=1}^{N} e^{-Q_i^2/2} \prod_{1 \le j < \ell \le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)(x_j - x_\ell)$

- Still an exact energy eigenstate (now highly excited) after switch $\gamma_{\mathrm{F}} = +\infty \rightarrow -\infty \Rightarrow$ *absolutely stable*.

Two sTG states of spinor fermions

Fermionic sTG gas:

- Change γ_{F} from $\gg 1$ to $\ll -1$ as in Innsbruck experiment for bosons. Exact ground state not known for finite γ_{F} in trapped case, but following sTG-ideal Fermi hybrid should be a good approximation:
  $\psi_{sTG} \approx [\prod_{1 \leq j < \ell \leq N} \beta(x_j, \sigma_j; x_\ell, \sigma_\ell)] \prod_{j=1}^{N} \exp\left(-\frac{x_j^2}{2x_{\mathrm{osc}}^2}\right)$
  where $\beta = (\delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \uparrow} + \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \downarrow})x_{j\ell} + (\delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \downarrow} - \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \uparrow})D_\nu(|x_{j\ell}/x_{\mathrm{osc}}|)e^{x_{j\ell}^2/4x_{\mathrm{osc}}^2}$
  This ψ_{sTG} satisfies contact conditions exactly.

Two sTG states of spinor fermions

Fermionic sTG gas:

- Change γ_{F} from $\gg 1$ to $\ll -1$ as in Innsbruck experiment for bosons. Exact ground state not known for finite γ_{F} in trapped case, but following sTG-ideal Fermi hybrid should be a good approximation:
  $\psi_{sTG} \approx [\prod_{1 \le j < \ell \le N} \beta(x_j, \sigma_j; x_\ell, \sigma_\ell)] \prod_{j=1}^{N} \exp\left(-\frac{x_j^2}{2x_{\text{osc}}^2}\right)$
  where $\beta = (\delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \uparrow} + \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \downarrow})x_{j\ell} + (\delta_{\sigma_j \uparrow}\delta_{\sigma_\ell \downarrow} - \delta_{\sigma_j \downarrow}\delta_{\sigma_\ell \uparrow})D_\nu(|x_{j\ell}/x_{\text{osc}}|)e^{x_{j\ell}^2/4x_{\text{osc}}^2}$
  This ψ_{sTG} satisfies contact conditions exactly.
- Reduces to TG-ideal Fermi ground state at $\gamma_{\text{F}} = +\infty$, still exact eigenstate at $\gamma_{\text{F}} = -\infty \Rightarrow$ Probability of transition to sTG gas of bound dimers under switch $\gamma_{\text{F}} \gg 1 \rightarrow \ll -1$ is $\ll 1$, contrary to Chen et al.

TG and sTG states of spin-1 bosons

- Model: Bosonic atoms in 1D trap in three different hyperfine states labelled as $\sigma = -1, 0, 1$, kinetic energy plus spin-independent harmonic trap potential plus spin-independent Lieb-Liniger (LL) delta interaction. Hamiltonian has same form as for spinor fermions:

$$\hat{H}_{\mathrm{B}} = \sum_{j=1}^{N} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2} x_j^2 \right) + g_{\mathrm{B}} \sum_{1 \le j < \ell \le N} \delta(x_j - x_\ell)$$

TG and sTG states of spin-1 bosons

- Model: Bosonic atoms in 1D trap in three different hyperfine states labelled as $\sigma = -1, 0, 1$, kinetic energy plus spin-independent harmonic trap potential plus spin-independent Lieb-Liniger (LL) delta interaction. Hamiltonian has same form as for spinor fermions:
$\hat{H}_{\mathrm{B}} = \sum_{j=1}^{N} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2} x_j^2 \right) + g_{\mathrm{B}} \sum_{1 \le j < \ell \le N} \delta(x_j - x_\ell)$
- Both 3D s-wave and p-wave scattering allowed. For TG and sTG one needs strong s-wave interaction due to 3D s-wave Feshbach resonance ⇒ 1D even-wave resonance ⇒ LL delta interaction with large g_{B}, with relative wave function also symmetric under spin exchange. Space-odd wave function cancels LL interaction ⇒ no odd-wave interaction.

TG and sTG states of spin-1 bosons

Exact solution for $N = 2$:

- $\gamma_{\mathrm{B}} = 2mg_{\mathrm{F}}/\hbar^2 n \to +\infty$: Ground state is space-even TG state, relative wave function $\phi(x) = (|x|/x_{\mathrm{osc}})e^{-(|x|/x_{\mathrm{osc}})^2/2}$, as for spinor Fermi gas. $\gamma_{\mathrm{F}} \to -\infty$: Still an exact energy eigenstate $\Rightarrow$ *completely stable* against collapse to McGuire's state.

TG and sTG states of spin-1 bosons

Exact solution for $N = 2$:

- $\gamma_{\mathrm{B}} = 2mg_{\mathrm{F}}/\hbar^2 n \to +\infty$: Ground state is space-even TG state, relative wave function $\phi(x) = (|x|/x_{\mathrm{osc}})e^{-(|x|/x_{\mathrm{osc}})^2/2}$, as for spinor Fermi gas. $\gamma_{\mathrm{F}} \to -\infty$: Still an exact energy eigenstate $\Rightarrow$ *completely stable* against collapse to McGuire's state.
- $-\infty < \gamma_{\mathrm{F}} \ll -1$: One solution same as for Bose sTG gas, $\phi(x) =$ const.$D_\nu(|x|/x_{\mathrm{osc}})$, node $|x| = x_{\mathrm{node}} > 0$, *metastable* against collapse (sTG).

TG and sTG states of spin-1 bosons

Exact solution for $N = 2$:

- $\gamma_{\mathsf{B}} = 2mg_{\mathsf{F}}/\hbar^2 n \to +\infty$: Ground state is space-even TG state, relative wave function $\phi(x) = (|x|/x_{\mathsf{osc}})e^{-(|x|/x_{\mathsf{osc}})^2/2}$, as for spinor Fermi gas. $\gamma_{\mathsf{F}} \to -\infty$: Still an exact energy eigenstate $\Rightarrow$ *completely stable* against collapse to McGuire's state.
- $-\infty < \gamma_{\mathsf{F}} \ll -1$: One solution same as for Bose sTG gas, $\phi(x) =$ const.$D_\nu(|x|/x_{\mathsf{osc}})$, node $|x| = x_{\mathsf{node}} > 0$, *metastable* against collapse (sTG).
- Another solution: bound boson dimer: $\gamma_{\mathsf{F}} = -\infty$: TG still exact eigenstate $\Rightarrow$ Transition probability to dimer under $\gamma_{\mathsf{B}} = +\infty \to -\infty$ *exactly zero*, still $\ll 1$ under $\gamma_{\mathsf{F}} \gg 1 \to \gamma_{\mathsf{F}} \ll -1$, as for spinor Fermi gas.

TG and sTG states of spin-1 bosons

Generalization to $N > 2$: M.D. Girardeau, arXiv:1008.0428

- Exact ground state in TG limit $\lambda_B \to +\infty$ generated from spinless ideal Fermi ground state ψ_{ideal} by space-spin antisymmetric mapping M: $\psi_B = M(x_1, \sigma_1; \cdots; x_N, \sigma_N)\psi_{\text{ideal}}$ where $M = \prod_{1 \le j < \ell \le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)$, $\alpha(x_j, \sigma_j; x_\ell, \sigma_\ell) = \sum_{s=-1}^{1} \delta_{\sigma_j, s}\delta_{\sigma_\ell, s}\text{sgn}(x_j - x_\ell) + \sum_{-1 \le s_1 < s_2 \le 1}(\delta_{\sigma_j, s_j}\delta_{\sigma_\ell, s_\ell} - \delta_{\sigma_j, s_\ell}\delta_{\sigma_\ell, s_j})$. Gives space-even, spin-even TG cusps.

TG and sTG states of spin-1 bosons

Generalization to $N > 2$: M.D. Girardeau, arXiv:1008.0428

- Exact ground state in TG limit $\lambda_B \to +\infty$ generated from spinless ideal Fermi ground state ψ_{ideal} by space-spin antisymmetric mapping M: $\psi_B = M(x_1, \sigma_1; \cdots; x_N, \sigma_N)\psi_{\text{ideal}}$ where $M = \prod_{1 \le j < \ell \le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)$, $\alpha(x_j, \sigma_j; x_\ell, \sigma_\ell) = \sum_{s=-1}^{1} \delta_{\sigma_j, s}\delta_{\sigma_\ell, s}\text{sgn}(x_j - x_\ell) + \sum_{-1 \le s_1 < s_2 \le 1}(\delta_{\sigma_j, s_j}\delta_{\sigma_\ell, s_\ell} - \delta_{\sigma_j, s_\ell}\delta_{\sigma_\ell, s_j})$. Gives space-even, spin-even TG cusps.
- Then TG-ideal Fermi gas hybrid ground state $\psi_{B0} = C_N[\prod_{i=1}^{N} e^{-Q_i^2/2}] \prod_{1 \le j < \ell \le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)(x_j - x_\ell)$

TG and sTG states of spin-1 bosons

Generalization to $N > 2$: M.D. Girardeau, arXiv:1008.0428

- Exact ground state in TG limit $\lambda_B \to +\infty$ generated from spinless ideal Fermi ground state ψ_{ideal} by space-spin antisymmetric mapping M: $\psi_B = M(x_1, \sigma_1; \cdots; x_N, \sigma_N)\psi_{\text{ideal}}$ where $M = \prod_{1\le j<\ell\le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)$, $\alpha(x_j, \sigma_j; x_\ell, \sigma_\ell) = \sum_{s=-1}^{1} \delta_{\sigma_j,s}\delta_{\sigma_\ell,s}\text{sgn}(x_j - x_\ell) + \sum_{-1\le s_1<s_2\le 1}(\delta_{\sigma_j,s_j}\delta_{\sigma_\ell,s_\ell} - \delta_{\sigma_j,s_\ell}\delta_{\sigma_\ell,s_j})$. Gives space-even, spin-even TG cusps.
- Then TG-ideal Fermi gas hybrid ground state $\psi_{B0} = C_N[\prod_{i=1}^{N} e^{-Q_i^2/2}] \prod_{1\le j<\ell\le N} \alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)(x_j - x_\ell)$
- Still exact energy eigenstate (now highly excited) after switch $\lambda_B = +\infty \to -\infty \Rightarrow$ *absolutely stable*.

TG and sTG states of spin-1 bosons

Generalization to $N > 2$: M.D. Girardeau, arXiv:1008.0428

- Exact ground state in TG limit $\lambda_B \to +\infty$ generated from spinless ideal Fermi ground state ψ_{ideal} by space-spin antisymmetric mapping M: $\psi_B = M(x_1, \sigma_1; \cdots; x_N, \sigma_N)\psi_{\text{ideal}}$ where $M = \prod_{1\le j<\ell\le N}\alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)$, $\alpha(x_j, \sigma_j; x_\ell, \sigma_\ell) = \sum_{s=-1}^{1}\delta_{\sigma_j,s}\delta_{\sigma_\ell,s}\text{sgn}(x_j - x_\ell) + \sum_{-1\le s_1<s_2\le 1}(\delta_{\sigma_j,s_j}\delta_{\sigma_\ell,s_\ell} - \delta_{\sigma_j,s_\ell}\delta_{\sigma_\ell,s_j})$. Gives space-even, spin-even TG cusps.
- Then TG-ideal Fermi gas hybrid ground state $\psi_{B0} = C_N[\prod_{i=1}^{N} e^{-Q_i^2/2}]\prod_{1\le j<\ell\le N}\alpha(x_j, \sigma_j; x_\ell, \sigma_\ell)(x_j - x_\ell)$
- Still exact energy eigenstate (now highly excited) after switch $\lambda_B = +\infty \to -\infty \Rightarrow$ *absolutely stable*.
- Generalization to finite $\lambda_B \ll -1$ is metastable sTG state. See the paper, also for 3^N-fold spin degeneracy.