Doppler Shift, Stellar Aberration, and Convection of Light by Moving Media

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The characteristics of a beam of light emanating from a source in uniform motion with respect to an observer differ from those measured when the source is stationary. In general, it is irrelevant whether the source is stationary and the observer in motion or vice versa; the observed characteristics depend only on the relative motion. The observed frequency of the light, for example, has been known to depend on this relative motion since the Austrian physicist Christian Doppler (1842) showed the effect to exist both for sound waves and light waves.[1]

The perceived direction of propagation of a light beam also depends on the relative motion of its source and the observer. The English astronomer James Bradley (1727) was the first to argue that the motion of the earth in its orbit around the sun causes a periodic shift of the apparent position of fixed stars as observed from the earth: a telescope viewing a star must be tilted in the direction of the earth's motion. Although this so-called stellar aberration could be explained on the basis of the corpuscular theory of light accepted at the time,[1] certain features of it remained poorly understood until the advent of Einstein's special theory of relativity in 1905.

The mid-nineteenth century measurements of the speed of light in moving media could be made to agree with the prevailing theories at the time only if one assumed that the moving medium partially carried the luminiferous ether, the hypothetical medium which filled the universe and in which the light waves propagated. The magnitude of this ether convection depended on the velocity as well as the refractive index of the moving medium.[1] Since the refractive index is wavelength dependent, implicit in these theories was the assumption that different ethers exist for different light colors, each being carried at a different rate by the moving medium.

This ad hoc and unsatisfactory state of affairs came to an end with the advent of the special theory of relativity. Doppler shift, stellar aberration, and the convection of light by moving media are the various manifestations of the same fundamental phenomenon: different relative perceptions of space and time for observers in motion with respect to one another. In this article we derive general formulas for all three phenomena by applying the Lorentz transformation to a plane electromagnetic wave. Examples will be used to clarify the physics behind the formulas.

Plane waves and the Lorentz transformation

A plane electromagnetic wave of frequency f propagating in free space along a direction specified by the polar and azimuthal angles (θ, ϕ) within a Cartesian XYZ coordinate system has the following form:

$$\boldsymbol{a}(x, y, z, t) = A_o \exp\{i2\pi(f/c)[(\sin\theta\cos\phi)\,x + (\sin\theta\sin\phi)\,y + (\cos\theta)\,z - ct]\}. \qquad (1a)$$

Here c is the speed of light in free space, and the complex vector A_o denotes the strength of the field at the origin of the coordinate system. Let us define the coordinates of a point in space-time as $p = (x, y, z, ict)$. The coefficients appearing in the exponent of the plane wave in Eq. (1a) can then be grouped together as $\sigma = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta, i]$, and the equation may be written in compact form,

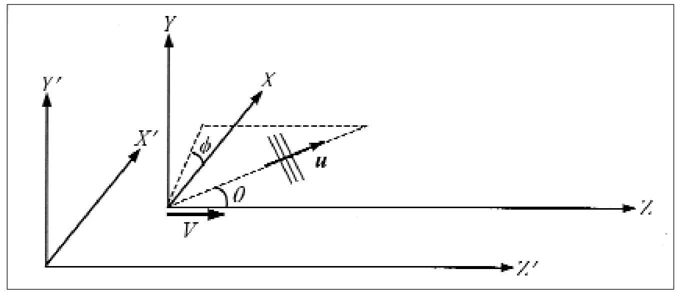

Figure 1. In the Cartesian XYZ coordinate system a plane wave of frequency f (wavelength = λ) propagates along the unit vector $\boldsymbol{u}$. The polar and azimuthal angles of $\boldsymbol{u}$ are denoted by θ and ϕ. As seen from another system, $X'Y'Z'$, the XYZ system moves at a constant velocity $\boldsymbol{V}$ along the Z-axis. From the perspective of an observer stationary in $X'Y'Z'$, the plane wave is Doppler shifted to a different frequency f', and the polar angle of its propagation direction has a different value θ'. The azimuthal angle ϕ, however, remains the same in the two systems.

$$\boldsymbol{a}(x, y, z, t) = A_o \exp[i2\pi(f/c)\,\sigma\, p^T], \qquad (1b)$$

where superscript T denotes a transposed vector. A second inertial system, $X'Y'Z'$, in which the XYZ system moves with uniform velocity V along the common Z-direction is shown in Fig. 1. The origins of the two systems coincide at $t = t' = 0$. The point $p' = (x', y', z', ict')$ in the new coordinate system is related to p by the Lorentz transformation,[1,2]

$$p^T = L\,p'^T, \qquad (2a)$$

where the 4×4 transformation matrix L is given by

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{1-(V/c)^2} & i(V/c)/\sqrt{1-(V/c)^2} \\ 0 & 0 & -i(V/c)/\sqrt{1-(V/c)^2} & 1/\sqrt{1-(V/c)^2} \end{pmatrix}. \qquad (2b)$$

(Recall that c is a constant, having the same value in any frame of reference in which it is measured.) L is a unitary matrix whose in-

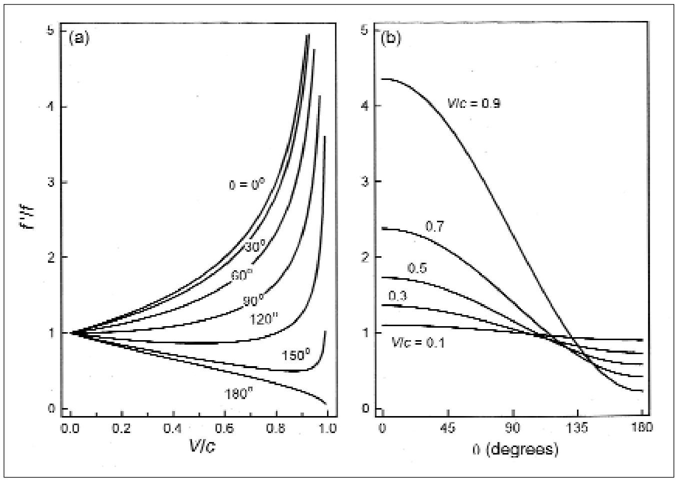

Figure 2. A plane wave of frequency *f* and propagation direction (θ, ϕ) in the *XYZ* coordinates is observed from the *X′Y′Z′* system of Fig. 1. The Doppler-shifted frequency *f′* seen by the observer is a function of *V* and θ, but does not depend on ϕ. (a) Plots of *f′/f* versus *V/c* for several values of θ. (b) Plots of *f′/f* versus θ for different values of *V/c*.

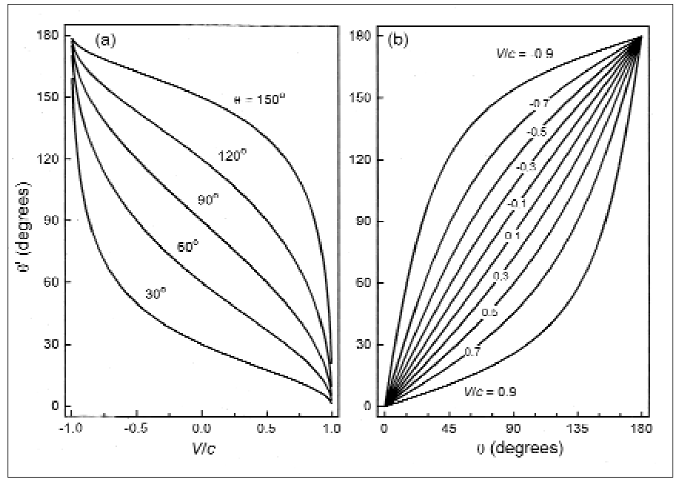

Figure 3. A plane wave of frequency *f* and propagation direction (θ, ϕ) in the *XY Z* coordinates is observed from the *X′Y′Z′* system of Fig. 1. The polar angle θ′ of the beam seen by the observer is a function of *V* and θ, but does not depend on ϕ. (a) Plots of θ′ versus *V/c* for several values of θ. (b) Plots of θ′ versus θ for different values of *V/c*.

verse is the same as its transpose, i.e., LL^T equals a 4 × 4 identity matrix. We substitute for p^T in Eq. (1b) from Eq. (2a), and evaluate σL as follows:

$$\sigma L = \frac{1 + (V/c)\cos\theta}{\sqrt{1-(V/c)^2}}[\sin\theta' \cos\phi, \sin\theta'\sin\phi, \cos\theta', i]. \quad (3a)$$

Here

$$\sin\theta' = \sqrt{1-(V/c)^2}\,\sin\theta \,/\, [1 + (V/c)\cos\theta]\,, \quad (3b)$$

$$\cos\theta' = (\cos\theta + V/c) \,/\, [1 + (V/c)\cos\theta]\,. \quad (3c)$$

It is readily verified from Eqs. (3b, 3c) that $\sin^2\theta' + \cos^2\theta' = 1$, which is needed if the above definition of θ′ is to be meaningful. We conclude that the plane wave in *X Y Z* remains a plane wave in *X′Y′Z′*, albeit with a different frequency and a different propagation direction.

Doppler shift

Replacing σp^T in Eq. (1b) with $\sigma L p'^T$ and using Eq. (3a), it becomes clear that the optical frequency *f′* of the plane wave as measured in the *X′Y′Z′* system is given by

$$f' = f\,[1 + (V/c)\cos\theta]\,/\sqrt{1-(V/c)^2}\,. \quad (4)$$

This is the relativistic formula for the Doppler shift,[2] valid for all propagation directions θ and all speeds *V*. When θ = 0°, the propagation direction and the motion of the observer are antiparallel. In this case *f′* is greater than *f* (i.e., blue shifted) according to the following formula:

$$f' = f\,\sqrt{[1 + (V/c)] \,/\, [1 - (V/c)]}\,. \quad (5a)$$

When θ = 180°, the propagation direction and the motion of the observer are parallel, in which case *f′* is less than *f* (i.e., red shifted) as follows:

$$f' = f\,\sqrt{[1 - (V/c)] \,/\, [1 + (V/c)]}\,. \quad (5b)$$

When θ = 90°, the observer is moving at right angles to the propagation direction. The classical analysis does not yield any Doppler shift in this case,[1] but the relativistic formula yields

$$f' = f/\sqrt{1-(V/c)^2}\,. \quad (5c)$$

It is possible to have a direction of propagation with no Doppler shift at all, i.e., *f′* = *f*. From Eq. (4) this direction is found to be:

$$\cos\theta = [\sqrt{1-(V/c)^2} - 1] \,/\, (V/c). \quad (6)$$

Substitution into Eqs. (3b, 3c) reveals that, when the above condition is satisfied, θ′ = 180° – θ.

Based on Eq. (4), Fig. 2(a) shows plots of *f′/f* versus *V/c* for several values of θ, while Fig. 2(b) shows plots of *f′/f* versus θ for different values of *V/c*. At a given velocity *V*, the beam is blue-shifted when θ = 0°, i.e., the observer moves opposite the direction of propagation, and red-shifted when θ = 180°, i.e., the observer moves along the propagation direction. If θ is varied continuously from 0° to 180°, the frequency changes from blue-shifted to red-shifted, becoming equal to *f* somewhere after θ = 90°. As *V* increases, the crossing point occurs at larger angles θ.

Stellar aberration

The direction of propagation of the beam perceived by the observer in the *X′Y′Z′* frame has polar angle θ′, given by Eqs. (3b, 3c), and azimuthal angle ϕ. Dividing Eq. (3b) by Eq. (3c) yields,[2]

$$\tan\theta' = \sqrt{1-(V/c)^2}\,\sin\theta \,/\, (\cos\theta + V/c). \quad (7)$$

Figure 3(a) shows plots of θ′ as function of *V/c* for several values of θ. Similarly, Fig. 3(b) shows plots of θ′ versus θ for different values of *V/c*. It is clear that, for a given θ, the apparent direction of

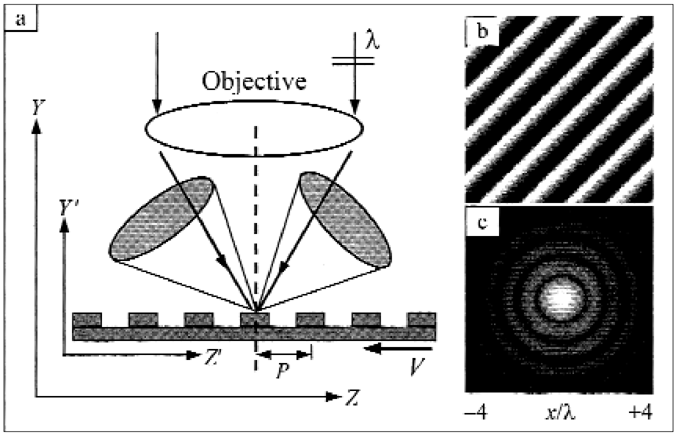

Figure 4. (a) A monochromatic plane wave of wavelength λ is focused, through an objective lens of numerical aperture NA, onto a diffraction grating. The period P of the grating is small enough to support only the 0^{th} and $\pm 1^{st}$ diffraction orders upon reflection. The three cones of light returning from the grating partially overlap in the exit pupil, giving rise to a "baseball pattern." When the grating moves at constant velocity $\boldsymbol{V}$ in the focal plane, the contrast within the baseball pattern shows a periodic oscillation. (b) Trapezoidal profile of a grating having period $P = 1.2\,\lambda$ and groove depth = 0.15λ. (c) Logarithmic plot of intensity distribution at the focal plane of a uniformly illuminated 0.6 NA objective.

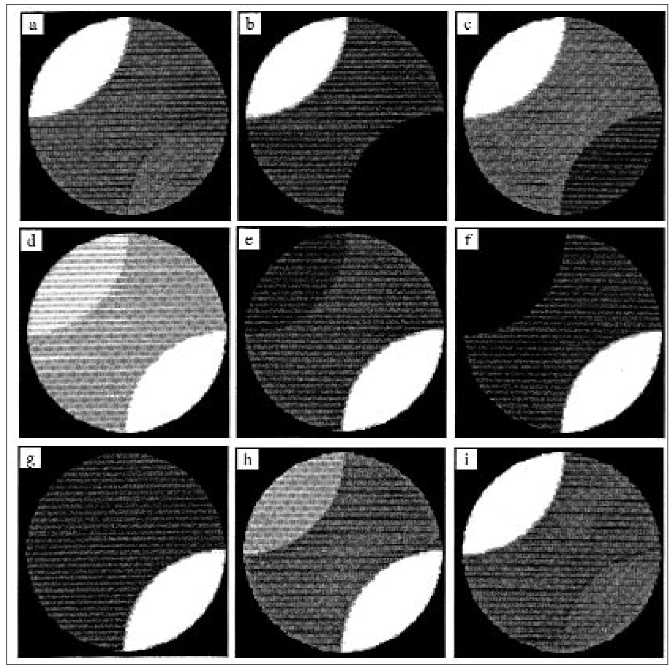

Figure 5. Patterns of intensity distribution observed at the exit pupil of the objective of Fig. 4. From (a) to (i) the distance between the groove center and the center of the focused spot is 0, 0.2, 0.4, 0.5, 0.6, 0.8, 1.0, 1.1, and 1.2 (in units of λ).

propagation depends on the relative velocity between the observer and the light source. For instance, if a telescope is aimed at a distant star far above the plane of the solar system, then in a reference frame where the star is stationary, $\theta = 90°$. However, for an earth-bound observer, $\cos\theta' = V/c$, where $V \approx 31$ km/s is the speed of the earth in its orbit around the sun. As the earth travels in its orbit, the direction of V changes and so does the apparent location of the star. In a six-month period, $\cos\theta'$ changes by $2V/c$, causing a shift of $\Delta\theta' \approx 0.012° \approx 43''$ in the star's apparent location. (In contrast, the size of the parallax for the nearest star as measured from the same location on the earth over a six month period is less than $2''$.)

Diffraction of light from a grating in uniform motion

Figure 4(a) shows a uniform beam of frequency f (and wavelength $\lambda = c/f$) focused onto a diffraction grating through an objective lens of numerical aperture NA. The grating's period P is sufficiently small to allow only the 0^{th}, -1^{st}, and $+1^{st}$ orders of diffraction to appear upon reflection. The three cones of light thus reflected from the grating are captured by the lens in the return path. The partial overlap of the diffracted cones at the lens's exit pupil (resulting in interference among them) gives rise to the so-called baseball pattern of intensity distribution. When the grating moves at a constant velocity V along the Z-axis, the contrast within the baseball pattern shows a periodic oscillation. This is caused by the variation of the relative phase between the $\pm 1^{st}$ and the 0^{th} diffracted orders, the phase being dependent on the position of the focused spot relative to the grooves on the grating.

As a specific example, Fig. 4(b) shows the surface profile of a grating with a trapezoidal cross section having period $P = 1.2\lambda$ and groove depth = 0.15λ. Figure 4(c), a logarithmic plot of the intensity distribution at the focal plane of a 0.6NA objective, shows the diameter of the central bright spot—the Airy disk—to be $1.22\lambda/\mathrm{NA} \approx 2\lambda$. Figure 5 shows computed patterns of reflected intensity at the exit pupil of the objective for several positions of the focused spot over the grating.[3] From (a) to (i), the groove center's distance from the center of the focused spot is 0, 0.2λ, 0.4λ, 0.5λ, 0.6λ, 0.8λ, λ, 1.1λ, and 1.2λ, respectively. In these simulations the grating is assumed to be stationary in its various positions relative to the lens.

An alternative (and physically more accurate) explanation of the baseball patterns of Fig. 5 may be based on the Doppler shift between the 0^{th} order and the $\pm 1^{st}$ order diffracted light cones depicted in Fig. 4. From the viewpoint of an observer in the grating's rest frame, the incident cone of light moves with velocity V along the Z-axis. This cone is a superposition of a multitude of plane waves of differing directions and frequencies. With reference to Fig. 6, consider a plane wave of frequency f and propagation direction (θ, ϕ) in the XYZ coordinate system in which the lens is stationary. This plane wave, when seen from the grating's rest frame, has frequency f' given by Eq. (4) and propagation direction (θ', ϕ) given by Eq. (7). For the 0^{th} order reflected plane wave, the frequency remains f' but the propagation direction becomes $(\theta', -\phi)$. Viewed from the rest frame of the lens, this reflected 0^{th} order beam has frequency f and propagation direction $(\theta, -\phi)$. Thus the 0^{th} order reflected cone—which is simply a superposition of various reflected 0^{th} order plane waves—seen by the lens is ignorant of the velocity V of the grating.

As for the $+1^{st}$ order beam, in the grating's rest frame the diffracted plane wave has frequency f' and propagation direction $(\theta'_{+1}, \phi'_{+1})$, where, in accordance with Bragg's law,

$$\cos\theta'_{+1} = \cos\theta' + (\lambda'/P), \tag{8a}$$
$$\sin\theta'_{+1}\cos\phi'_{+1} = \sin\theta'\cos\phi\,. \tag{8b}$$

Back in the rest frame of the lens, the diffracted $+1^{st}$ order plane wave appears to have a new frequency f_{+1}, where

$$\Delta f = f_{+1} - f = -V / [P\sqrt{1 - (V/c)^2}] \quad (9)$$

is independent of the incident beam's propagation direction (θ, ϕ). The period P of the grating thus appears to have been foreshortened by the Lorentz contraction factor, and the Doppler shift Δf is proportional to the velocity V and inversely proportional to the (contracted) grating period. Since Δf is independent of the direction of the incident plane wave, the entire $+1^{st}$ order cone will be Doppler shifted by the same amount. This Doppler shift causes a beating at the exit pupil between the 0^{th} order and the $+1^{st}$ order cones in their area of overlap. The beat period, $1/\Delta f$, is independent of the groove profile as well as the NA of the lens. The same arguments apply to the -1^{st} order light cone, except that the Doppler shift in this case is $-\Delta f$.

We mention in passing that, for the plane wave incident at (θ, ϕ) in the rest frame of the lens, the propagation directions of the $\pm 1^{st}$ order reflected plane waves are ($\theta_{\pm 1}$, $\phi_{\pm 1}$), where

$$\cos\theta_{\pm 1} = \frac{\cos\theta \pm \lambda / [P\sqrt{1 - (V/c)^2}]}{1 \mp (V/c)\,\lambda / [P\sqrt{1 - (V/c)^2}]} \quad (10)$$

and $\phi_{\pm 1} = \phi'_{\pm 1}$ (see Eq. (8b)). Aside from the Lorentz contraction of the grating period P, there is a relativistic correction to Bragg's law of diffraction from a moving grating. This correction term, which appears in the denominator on the right-hand side of Eq. (10), is of the first-order in V/c.

Rayleigh range of a moving Gaussian beam

Figure 7 shows a Gaussian beam of wavelength λ propagating along the Z-axis in the XYZ coordinate system, in which the source of the beam is at rest. The beam diameter W_o at the waist increases by a factor of $\sqrt{2}$ at a distance $L_o = W_o^2/\lambda$, the Rayleigh range of the beam.[4] To an observer at rest in the $X'Y'Z'$ frame, the source of the Gaussian beam moves at constant velocity V along the Z-axis. Since the beam diameter at any cross-section is a measurable quantity along Y', from the perspective of the observer the beam diameters at the waist and at the Rayleigh range remain W_o and $\sqrt{2}W_o$, respectively. However, the distance L'_o between these two points appears to have shrunk by Lorentz contraction, that is, $L'_o = L_o\sqrt{1-(V/c)^2}$. At the same time, the observer perceives the wavelength λ of the beam to have shifted in accordance with the Doppler formula to a lower or higher value, depending on whether the motion of the light source is towards or away from the observer. The Gaussian beam formula then yields $L'_o = W_o^2/\lambda'$, so that, in the observer's rest frame, the Rayleigh range L'_o could be greater or less than L_o depending on whether the observer moves towards or away from the light source. The two conclusions thus reached are contradictory, leaving one to wonder which prediction, if any, will be borne out by experiment.

Upon closer inspection, the Gaussian beam will be recognized as a superposition of many plane waves of differing propagation directions. In the rest frame of the beam's source, all these plane waves have the same wavelength λ, but viewed from a moving frame the wavelengths differ for different propagation directions. (The propagation directions of the various plane waves are not the same in the two coordinate systems, thus resulting in a wider or narrower spatial frequency spectrum in $X'Y'Z'$ depending on the direction of motion of the observer.) The formula relating the Rayleigh range to the beam waist and to the wavelength has been derived with the implicit assumption that the beam is a superposition of plane waves of *identical* frequency.[4] This is true in the rest frame of the beam's source, but decidedly false in the moving frame. Therefore, our second method of determining L'_o must be wrong, leaving the Lorentz contracted L_o as the correct answer.

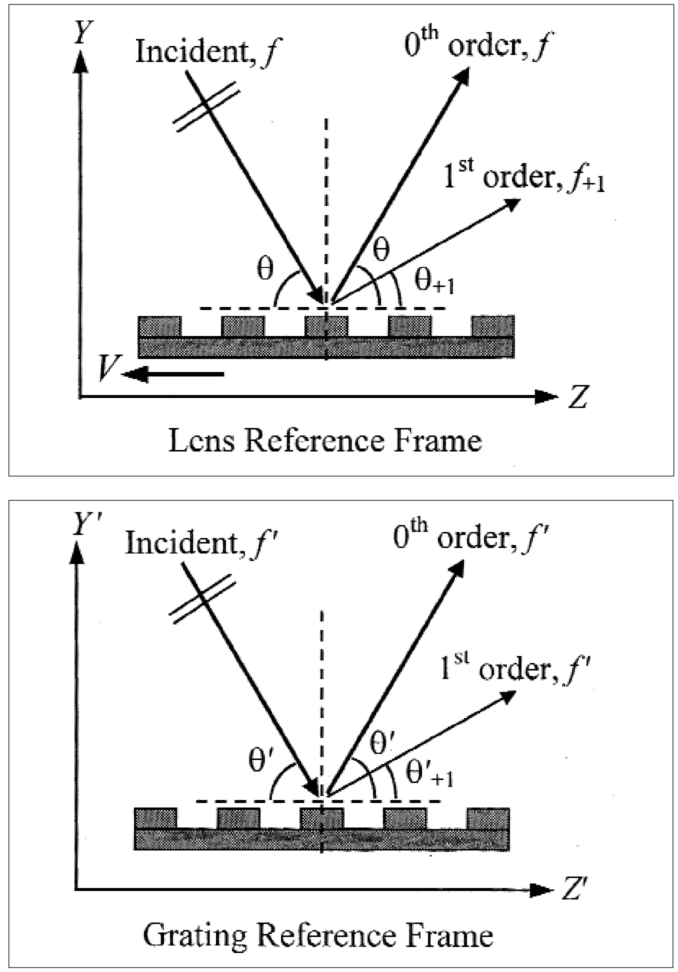

Figure 6. In the reference frame of the lens of Fig. 4, a plane wave of frequency f incident at an angle θ on a moving grating gives rise to a 0^{th} order diffracted beam of the same frequency and polar angle. The $+1^{st}$ order diffracted beam, however, will have frequency f_{+1} and polar angle θ_{+1}. In the grating's rest frame, the incident beam has frequency f' and polar angle θ'. All diffracted orders have the same frequency f'. The polar angle of the 0^{th} order beam is θ', while that of the $+1^{st}$ order beam is θ'_{+1}.

Convection of light by moving media

Consider a plane wave of frequency f and propagation direction (θ, ϕ), propagating in a stationary medium of refractive index n. By definition, the speed of light in this medium is c/n. The expression for the field distribution throughout space and time is similar to that given by Eq. (1a), namely,

$$\boldsymbol{a}(x, y, z, t) = A_o \exp\{i2\pi(f/c)[n(\sin\theta\cos\phi)x + n(\sin\theta\sin\phi)y + n(\cos\theta)z - ct]\}. \quad (11)$$

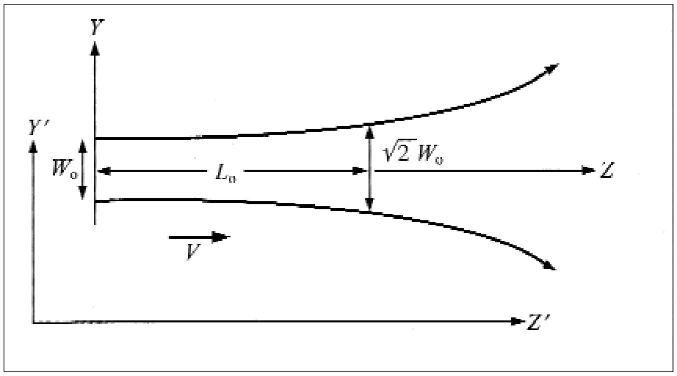

Figure 7. A Gaussian beam of wavelength λ propagates along the Z-axis. The beam diameter is W_o at the waist and $\sqrt{2}W_o$ at the Rayleigh range, which is a distance L_o from the waist. To an observer moving with constant velocity along the Z-axis, the beam diameters at the waist and at the Rayleigh range remain W_o and $\sqrt{2}W_o$, but the distance between them appears to have shrunk by the Lorentz contraction factor.

From the perspective of an observer whose frame of reference $X'Y'Z'$ is in uniform motion relative to the stationary medium (see Fig. 1), the expression for the field is obtained by substituting in Eq. (11) the Lorentz transformation of Eq. (2). This yields the same expression as in Eq. (11) with φ remaining the same but *f*, *n*, and θ changing as follows:

$$f' = f[1 + n(V/c)\cos\theta]\,/\sqrt{1-(V/c)^2}\ , \tag{12a}$$

$$n' = \left\{1 + \frac{(n^2-1)[1-(V/c)^2]}{[1+n(V/c)\cos\theta]^2}\right\}^{1/2}, \tag{12b}$$

$$\tan\theta' = n\sqrt{1-(V/c)^2}\,\sin\theta\,/\,[n\cos\theta + (V/c)]\ \cdot \tag{12c}$$

It is clear that the Doppler-shifted frequency f' and the polar angle θ' are not only functions of θ and V/c, as before, but they also depend on the refractive index *n* of the propagation medium. Similarly, the apparent index n' of the moving medium depends on *n*, V/c, and θ in accordance with Eq. (12b). For water of refractive index $n = 1.33$, Fig. 8(a) shows plots of n' versus V/c for several values of θ, while Fig. 8(b) shows plots of n' versus θ for different values of V/c.

In the special case when the beam moves in the same direction as the medium, $\theta = 0°$ and Eq. (12b) simplifies to

$$n' = \frac{n + (V/c)}{|1 + n(V/c)|}\ \cdot \tag{13}$$

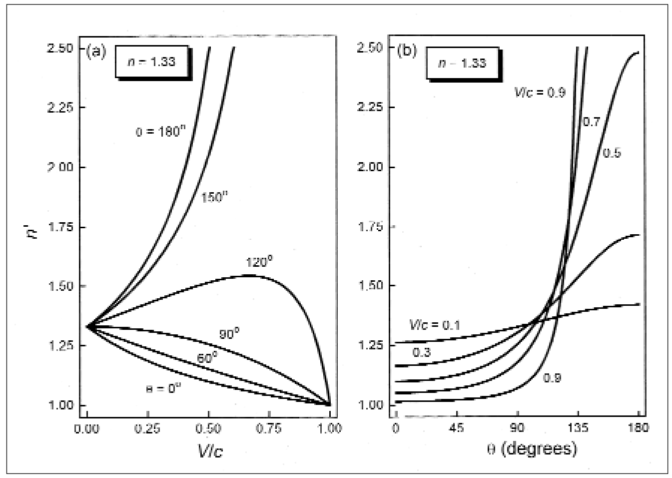

Figure 8. The refractive index n' of water ($n = 1.33$), moving at constant velocity *V* along the Z-axis, depends on *V* and on the propagation direction θ. In the water's rest frame, the assumed propagation direction of a plane wave of wavelength λ has polar and azimuthal angles (θ, φ). (a) Plots of n' versus V/c for several values of θ. (b) Plots of n' versus θ for different values of V/c.

When $V << c$, Eq. (13) yields Fresnel's formula[1] for the drag of light by a moving medium,

$$c/n' \approx (c/n) + (1 - n^{-2})V. \tag{14}$$

Thus the speed of light in a moving medium increases by a fraction of *V* if the light and the medium move in the same direction, whereas the apparent speed of light decreases when the light moves opposite the direction of motion of the medium.

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