Appendix B: MATHEMATICAL REVIEW

Analysis of physical optics phenomena often involves the use of the Fourier transform. This section provides a brief summary of some important relationships involving Fourier transforms. In general, the Fourier transform provides a linear relationship between a function of variables in one domain to an equivalent function in a second domain. For example, a function of time can be decomposed in to a collection of sinusoidal frequencies. The sinusoids are *basis functions* of the transformation. In optics, a useful mapping is from the *spatial* domain to the *spatial frequency* domain, where the basis functions are complex exponentials that can be interpreted as plane waves traveling at various angles in space. The details of these mappings are more fully developed in Chapter 5, but both the temporal and spatial Fourier mapping operations have the same mathematical foundation. In this section, basic properties of the Fourier transform and elementary Fourier transform pairs are meant to serve as a quick reference. A more complete development of Fourier mathematics can be found in various reference materials.[1,2,3,4]

B.1 Definitions:

One-dimensional Fourier transform:

$$\mathbf{F}_{\xi}[g(x)] = G(\xi) = \int_{-\infty}^{\infty} g(x)e^{-j2\pi\xi x}dx \tag{B.1}$$

Two-dimensional Fourier transform:

$$\mathbf{F}_{\eta}\mathbf{F}_{\xi}[g(x, y)] = G(\xi, \eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)e^{-j2\pi(\xi x + \eta y)}dxdy \tag{B.2}$$

One-dimensional inverse Fourier transform:

$$\mathbf{F}_{x}^{-1}[G(\xi)] = g(x) = \int_{-\infty}^{\infty} G(\xi)e^{j2\pi\xi x}d\xi \tag{B.3}$$

Two-dimensional inverse Fourier transform:

$$\mathbf{F}_{y}^{-1}\mathbf{F}_{x}^{-1}[G(\xi, \eta)] = g(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi, \eta)e^{j2\pi(\xi x + \eta y)}d\xi d\eta \tag{B.4}$$

1. J. W. Goodman, Introduction to Fourier Optics, 2nd Ed., McGraw-Hill, San Fransisco (1996)
2. R. Bracewell, The Fourier Transform and Its Application, McGraw-Hill, New York (1978)
3. J. Gaskill, Linear Systems, Fourier Transforms, and Optics, John Wiley and Sons, New York (1978)
4. http://mathworld.wolfram.com/FourierTransform.html

Note that the subscript of the bold capital letter indicates the transform variable.

B.2 Properties of Fourier transforms:

The following are some of the important properties of Fourier transforms. The proofs of these properties can be found in Appendix A of Goodman's Introduction to Fourier Optics, 2nd Edition.

B.2.1 Linearity:

$$\mathbf{F}_\xi[\alpha g(x) + \beta h(x)] = \alpha \mathbf{F}_\xi[g(x)] + \beta \mathbf{F}_\xi[h(x)] \quad \text{(B.5)}$$

B.2.2 Similarity:

If the Fourier transform of $g(x, y)$ is given by $\mathbf{F}_\eta \mathbf{F}_\xi[g(x, y)] = G(\xi, \eta)$, then

$$\mathbf{F}_\eta \mathbf{F}_\xi[g(ax, by)] = \frac{1}{|ab|} G\left(\frac{\xi}{a}, \frac{\eta}{b}\right). \quad \text{(B.6)}$$

B.2.3 Shifting property:

If $\mathbf{F}_\eta \mathbf{F}_\xi[g(x, y)] = G(\xi, \eta)$, then

$$\mathbf{F}_y \mathbf{F}_x[g(x - a, y - b)] = G(\xi, \eta) e^{-j2\pi(\xi a + \eta b)}. \quad \text{(B.7)}$$

B.2.4 Rayleigh's property (Parseval's theorem):

If $\mathbf{F}_\eta \mathbf{F}_\xi[g(x, y)] = G(\xi, \eta)$, then

$$\int_{\infty}^{-\infty} \int_{\infty}^{-\infty} |g(x, y)|^2 dx dy = \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} |G(\xi, \eta)|^2 d\xi d\eta. \quad \text{(B.8)}$$

The integral on the left-hand side of this theorem can be interpreted as the total energy contained in the waveform $g(x, y)$, whereas the integral on the right-hand side signifies the energy in the frequency domain. Thus, Rayleigh's property states that the total energy in the spatial domain is equivalent to the total energy in the frequency domain. In other words, energy is conserved (Parseval's theorem).

B.2.5 Convolution theorem:

If $\mathbf{F}_\eta \mathbf{F}_\xi[g(x, y)] = G(\xi, \eta)$, and $\mathbf{F}_\eta \mathbf{F}_\xi[h(x, y)] = H(\xi, \eta)$ then

$$\mathbf{F}_\eta\mathbf{F}_\xi[g(x,y)**h(x,y)] = \mathbf{F}_\eta\mathbf{F}_\xi\left[\iint_{-\infty} g(\alpha,\beta)h(x-\alpha,y-\beta)(d\alpha)d\beta\right] \tag{B.9}$$
$$= G(\xi,\eta)H(\xi,\eta)\ .$$

B.2.6 Autocorrelation theorem:

If $\mathbf{F}_\eta\mathbf{F}_\xi[g(x,y)] = G(\xi,\eta)$, then

$$\mathbf{F}_\eta\mathbf{F}_\xi[g(x,y)\star\star g^*(x,y)] = \mathbf{F}_\eta\mathbf{F}_\xi\left[\iint_{-\infty}^{\infty} g(\alpha,\beta)g^*(\alpha-x,\beta-y)d\alpha d\beta\right] \tag{B.10}$$
$$= |G(\xi,\eta)|^2\ ,$$

where the superscript asterisk (*) denotes complex conjugate.

Similarly,

$$\mathbf{F}_\eta\mathbf{F}_\xi[|g(x,y)|^2] = \iint_{-\infty}^{\infty} G(\alpha,\beta)G^*(\alpha-\xi,\beta-\eta)d\alpha d\beta\ . \tag{B.11}$$

B.2.7 Fourier integral theorem:

At each point of the continuity of $g(x,y)$,

$$\mathbf{F}_y\mathbf{F}_x\mathbf{F}_\eta^{-1}\mathbf{F}_\xi^{-1}[g(x,y)] = \mathbf{F}_y^{-1}\mathbf{F}_x^{-1}\mathbf{F}_\eta\mathbf{F}_\xi[g(x,y)] = g(x,y)\,. \tag{B.12}$$

At the points of discontinuity of $g(x,y)$, the two successive Fourier transforms will give the average value of $g(x,y)$ in a small neighborhood of that point.

B.2.8 Fourier transform of separable functions:

A function of two independent variables is *separable* with respect to a specific coordinate system if it can be written as a product of two functions, each of which depends only on one of the independent variables. Thus, the function $g(x,y)$ is separable in rectangular coordinates (x,y) if

$$g(x,y) = g_X(x)\cdot g_Y(y)$$

and similarly, in polar coordinates (r,θ) if

$$g(r,\theta) = g_R(r)\cdot g_\Theta(\theta)$$

Separable functions are easier to handle, in that in most cases separability often allows all two-dimensional manipulations to be reduced as a product of two one-dimensional manipulations. If the function $g(x,y)$ is separable, then its Fourier transform is given by

$$\begin{aligned} \mathbf{F}_\eta \mathbf{F}_\xi [g(x, y)] &= \iint_{-\infty}^{\infty} g(x, y) e^{-j2\pi(\xi x + \eta y)} dx dy \\ &= \int_{-\infty}^{\infty} g_X(x) e^{-j2\pi\xi x} dx \int_{-\infty}^{\infty} g_Y(y) e^{-j2\pi\eta y} dy \\ &= \mathbf{F}_\xi [g_X(x)] \mathbf{F}_\eta [g_Y(y)] \ . \end{aligned} \tag{B.13}$$

B.3 Circularly symmetric functions:

A function $g(x, y)$ is said to be circularly symmetric in polar coordinates (r, θ) if $g(r, \theta) = g_R(r)$. The Fourier transform of such special functions can be conveniently simplified, as shown below.

B.3.1 Fourier transform of circularly symmetric functions:

The Fourier transform of $g(x, y)$ in a rectangular coordinate system is given by

$$G(\xi, \eta) = \iint_{-\infty} g(x, y) e^{-j2\pi(\xi x + \eta y)} dx dy \ .$$

Transformation of the rectangular coordinates into polar coordinates is accomplished using the following relationships and definitions:

$r = \sqrt{x^2 + y^2}$, $y = r \sin\theta$, $x = r \cos\theta$

$\theta = \operatorname{atan}\left(\frac{y}{x}\right)$, $y = r \sin\theta$

$\rho = \sqrt{\xi^2 + \eta^2}$, $\xi = \rho \cos\theta$

$\phi = \operatorname{atan}\left(\frac{\xi}{\eta}\right)$, $\eta = \rho \sin\theta$

Let the Fourier transform of $g(x, y)$ in polar coordinates be given by $G(\rho, \phi)$. Applying the above transformation to the Fourier transform of circularly symmetric function $g(x, y) = g_R(r)$ in rectangular coordinates, we have

$$G(\rho, \phi) = \int_0^{2\pi} d\theta \int_0^{\infty} dr r g_R(r) e^{-j2\pi r \rho (\cos\theta \cos\phi + \sin\theta \sin\phi)}$$

$$G(\rho, \phi) = \int_0^{2\pi} d\theta \int_0^{\infty} dr r g_R(r) e^{-j2\pi r \rho \cos(\theta - \phi)},$$

or equivalently,

$$G(\rho, \phi) = G_P(\rho) = 2\pi \int_0^{\infty} g_R(r) J_0(2\pi r \rho) r dr = \mathbf{B}_\rho [g_R(r)]. \tag{B.14}$$

where J_0 (a) is the Bessel function of the first kind (zero order), where $J_0(a) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ja\cos(\theta - \phi)} d\theta$.

Thus, we see that the Fourier transform of a circularly symmetric function is itself circularly symmetric. This particular Fourier transform is referred to as the **Fourier-Bessel transform**, or **Hankel transform of zero order.**[5]

Similarly, the inverse Fourier transform of a circularly symmetric spectrum $G_P(\rho)$ is

$$g_R(r) = 2\pi\int_0^{\infty} G_P(\rho)J_0(2\pi r\rho)\rho d\rho = \mathbf{B}_r^{-1}[G_P(\rho)]. \tag{B.15}$$

B.3.2 Properties of the Fourier-Bessel transform:

This Fourier-Bessel transform is equivalent to a two-dimensional Fourier transform of a radially symmetric function. Thus, any property of Fourier transforms has an equivalent counterpart in Hankel transforms. Using **B** [ ] to represent the Fourier-Bessel transform, it follows from the Fourier integral theorem that at the points of continuity of $g_R(r)$

$$\mathbf{B}_r\mathbf{B}_\rho^{-1}[g_R(r)] = \mathbf{B}_r^{-1}\mathbf{B}_\rho[g_R(r)] = \mathbf{B}_r\mathbf{B}_\rho[g_R(r)] = g_R(r)\,. \tag{B.16}$$

Using the similarity theorem, it can be shown that,

$$\mathbf{B}_\rho[g_R(ar)] = \frac{1}{|a|^2}G_P\left(\frac{\rho}{a}\right). \tag{B.17}$$

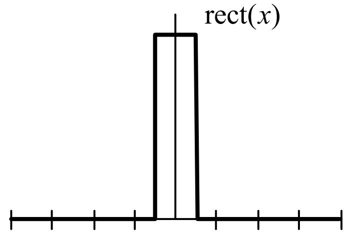
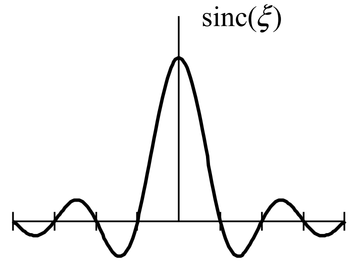
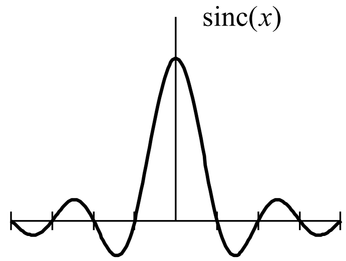
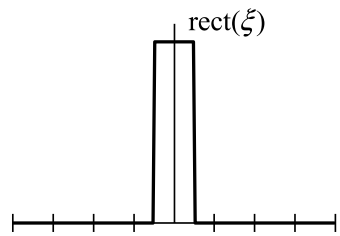
B.4 Functions and their Fourier transforms

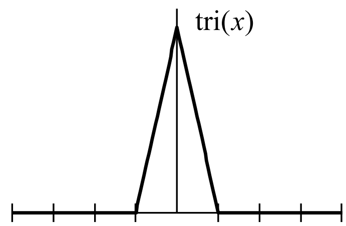
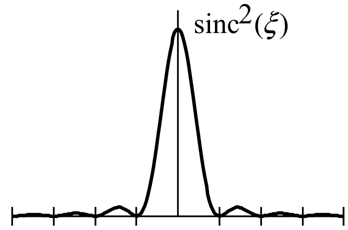
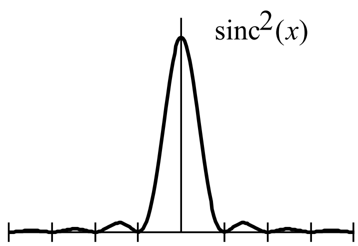
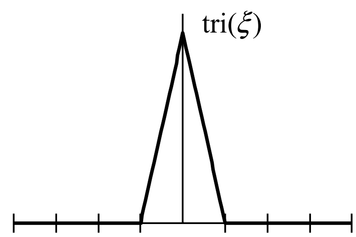
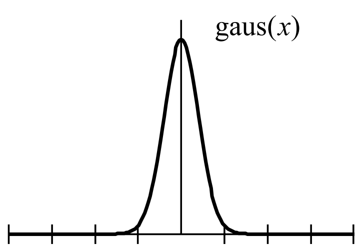
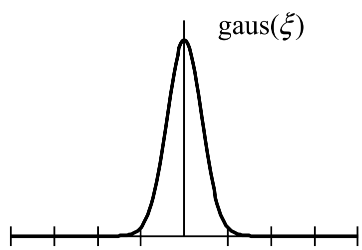
The following sections provide a few function definitions and their corresponding Fourier transforms. Section B.4.1 lists one-dimensional functions. section B.4.2 lists non-separable two-dimensional functions.

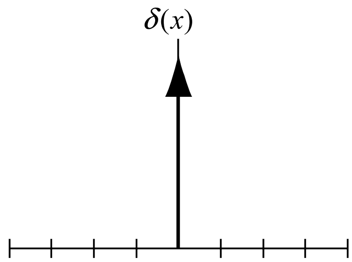
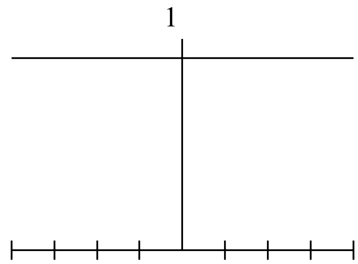
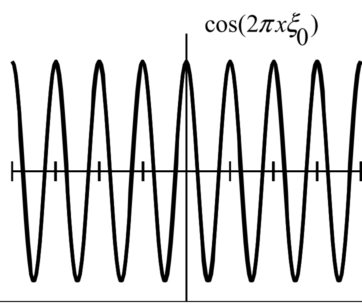
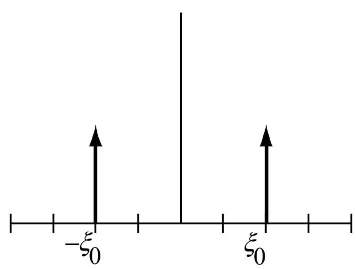
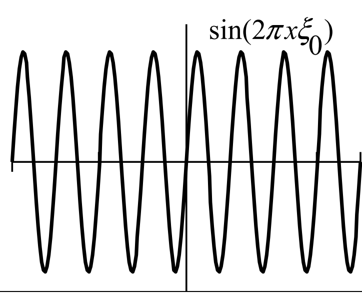
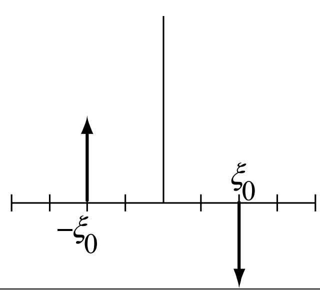
B.4.1 One-dimensional functions and their Fourier transforms

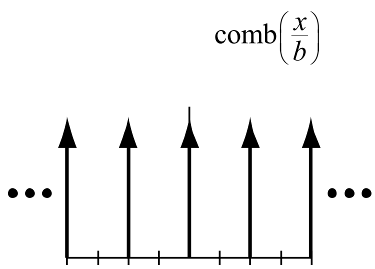
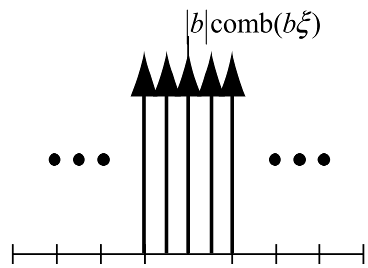
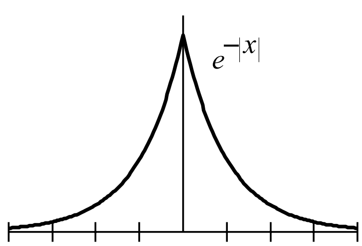
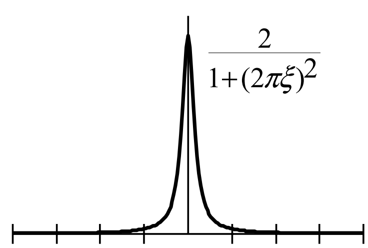
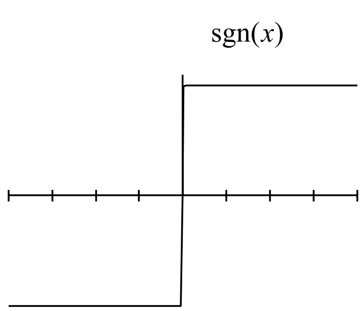
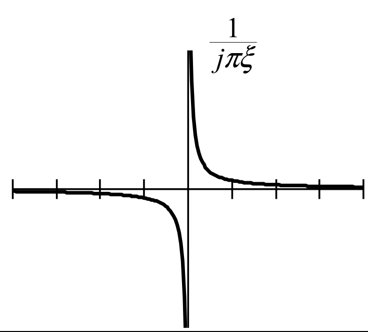
Several one-dimensional functions and their Fourier transforms are listed in Table B.1. These functions include the rectangle, sinc, gaus, delta, triangle, sinc square, sine, cosine, comb, decaying exponential, Lorentzian and sgn and it's Fourier transform. The shifted and scaled function $\frac{1}{b}\mathrm{rect}\left(\frac{x-a}{b}\right)$ is shown in Figure B.1. Detailed graphs of sinc(x), $\mathrm{sinc}^2(x)$, gaus(x), decaying exponential and Lorentzian are shown in Figures B.2 through B.5, respectively.

5. Read more about Hankel transforms at http://mathworld.wolfram.com/HankelTransform.html

Table B.1 One-dimensional functions and their Fourier transforms	
$g(x)$	$\mathbf{F}_{\xi}[g(x)]$
Rectangle $\mathrm{rect}(x)=\prod(x)=\begin{cases}1 & \lvert x\rvert<\frac{1}{2}\\ \frac{1}{2} & \lvert x\rvert=\frac{1}{2}\\ 0 & \text{otherwise}\end{cases}$ 	$\mathrm{sinc}(x)$ 
Sinc $\mathrm{sinc}(x)=\frac{\sin(\pi x)}{\pi x}$ 	$\mathrm{rect}(x)$ 

<table>
<tr><th>$g(x)$</th><th>$\mathbf{F}_{\xi}[g(x)]$</th></tr>
<tr><td>Triangle
$\text{tri}(x) = \Lambda(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$

</td><td>$\text{sinc}^2(x)$

</td></tr>
<tr><td>Sinc squared
$\text{sinc}^2(x) = \dfrac{\sin^2(\pi x)}{(\pi x)^2}$

</td><td>$\text{tri}(x)$

</td></tr>
<tr><td>Gaussian
$\text{gaus}(x) = e^{-\pi x^2}$

</td><td>$e^{-\pi \xi^2}$

</td></tr>
</table>

$g(x)$	$\mathbf{F}_\xi[g(x)]$
Delta function $\delta(x) = \lim_{b \to 0}\left[\frac{1}{\lvert b \rvert}\,\mathrm{gaus}\left(\frac{x}{b}\right)\right]$ 	1 
Cosine $\cos(2\pi x\xi_0)$ 	$\frac{1}{2}\left[\delta(\xi+\xi_0)+\delta(\xi-\xi_0)\right]$ 
Sine $\sin(2\pi x\xi_0)$ 	$\frac{1}{2j}[\delta(\xi+\xi_0)-\delta(\xi-\xi_0)]$ 

$g(x)$	$\mathbf{F}_\xi[g(x)]$
Comb function $\text{comb}\frac{x}{b} = \text{III}\left(\frac{x}{b}\right) = \lvert b\rvert \sum_{n=-\infty}^{\infty} \delta(x-nb)$ 	$\lvert b\rvert comb(b\xi)$ 
Decaying function $e^{-\lvert x\rvert}$ 	$\frac{2}{1+(2\pi\xi)^2}$ 
Signum function $\text{sgn}(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$ 	$\frac{1}{j\pi\xi}$ 

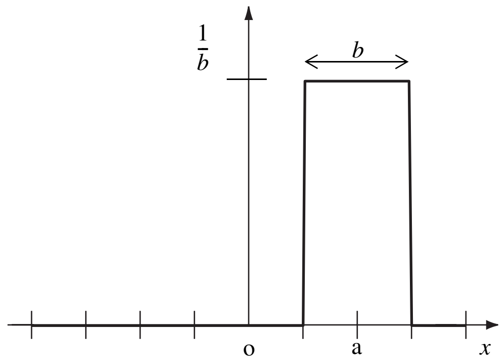


Figure B.1

B.4.2 Non-separable two-dimensional functions and their Fourier transforms

Two important functions are circ(r) and somb(r). Definitions of these functions are given in Table B.2. circ(r) is sometimes written as cyl(r). These functions are circularly symmetric and are functions of r (or ρ) only. The circ(r) and somb(r) functions form a Fourier transform pair, where

$$\mathbf{B}[\mathrm{circ}(r)] = \frac{\pi}{4}\mathrm{somb}(\rho)\,. \tag{B.18}$$

The circ(r) function resembles a pillbox, and the somb(r) function is similar to the sinc(x) function, except the zeros are not at integer values of r and the extrema have lower magnitudes. Detailed graphs of the somb(r) and $\mathrm{somb}^2(r)$ functions are shown in Figures B.6(a) and B.6(b), respectively. A table of zeros for the somb(r) function is given in Table B.3.

Table B.2 Non-Separable Two-Dimensional Functions
$\mathrm{circ}(r)=\mathrm{cyl}(r)=\begin{cases}1 & 0\le r\le\frac{1}{2}\\ \frac{1}{2} & r=\frac{1}{2}\\ 0 & \text{otherwise}\end{cases}$
$\mathrm{somb}(r)=\dfrac{2J_1(\pi r)}{\pi r}$

Table B.3 Zeros of the somb(r)function	
Zero	r
1^{st}	1.2196
2^{nd}	2.2331
3^{rd}	3.2383

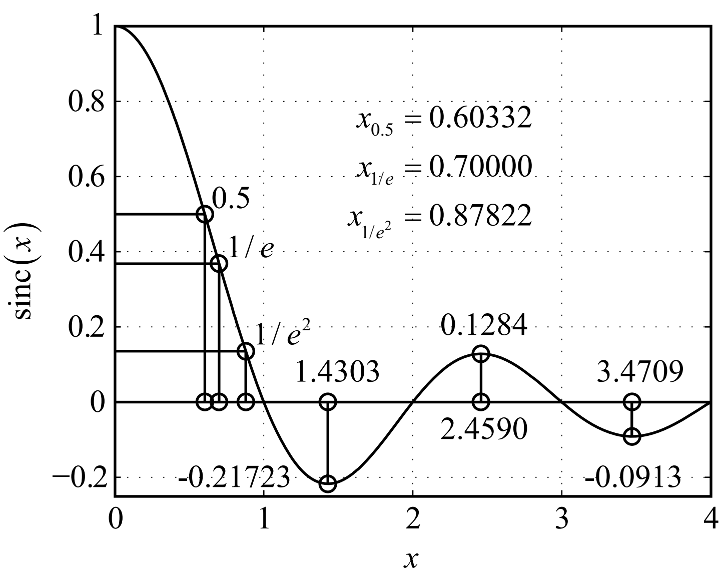


Figure B.2(a)

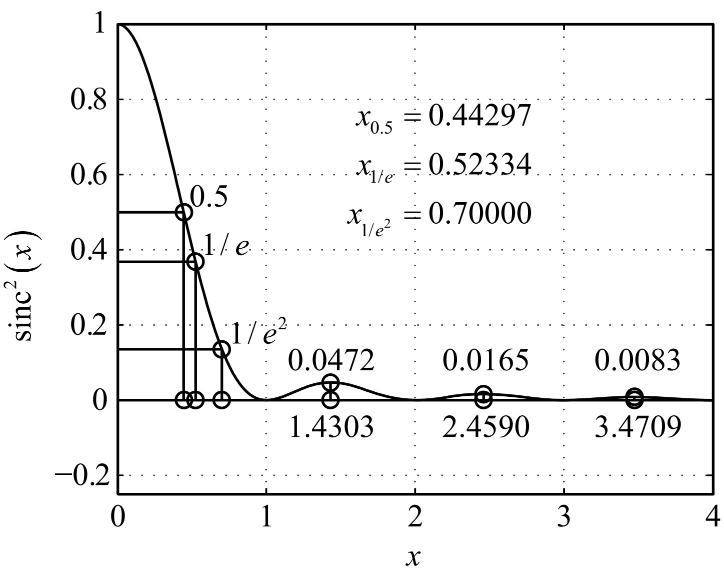


Figure B.2(b)

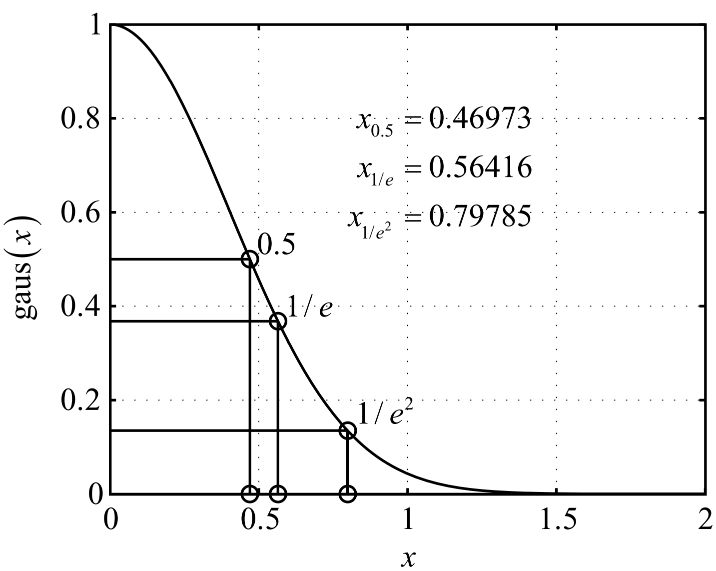


Figure B.3

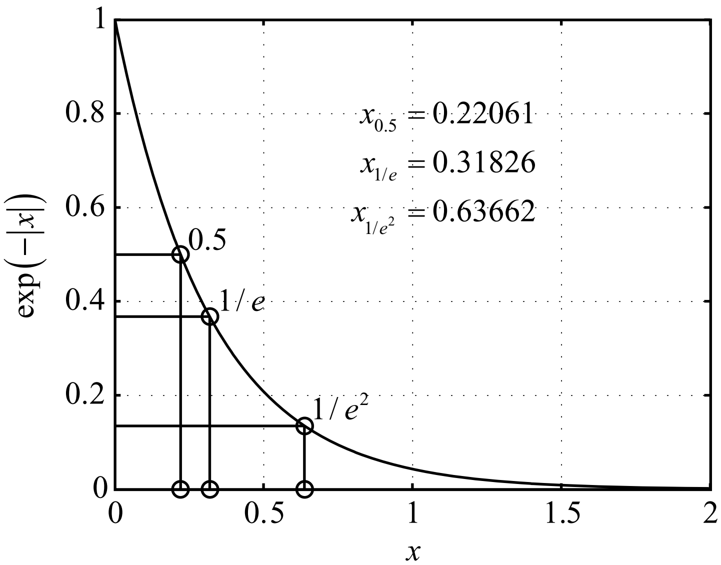


Figure B.4

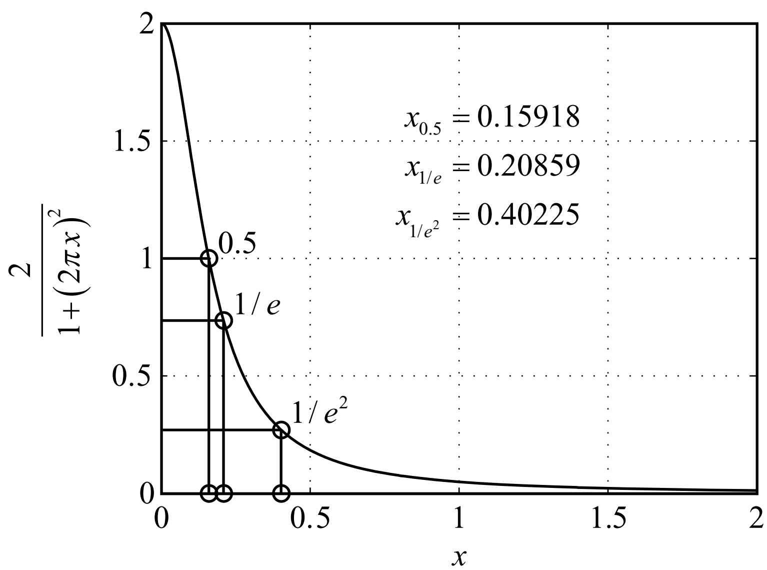


Figure B.5

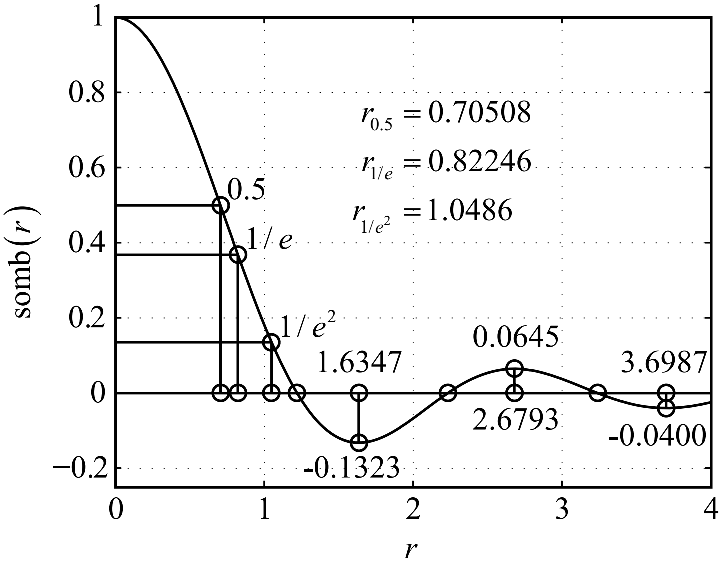


Figure B.6(a)

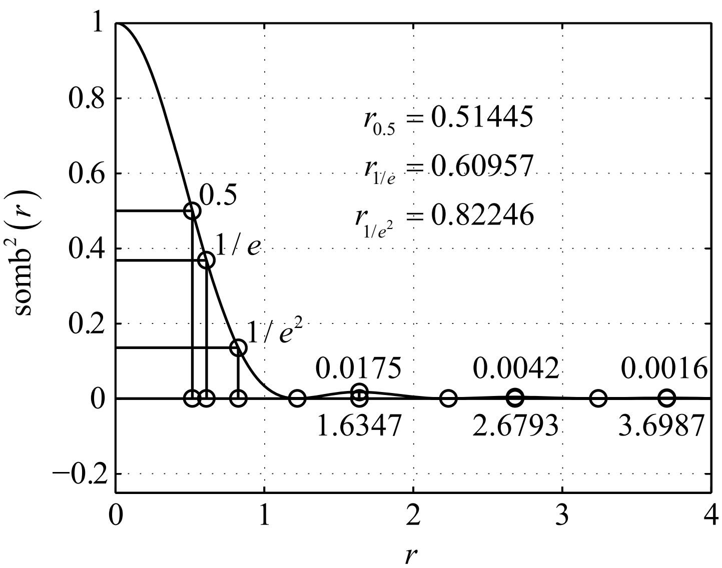


Figure B.6(b)