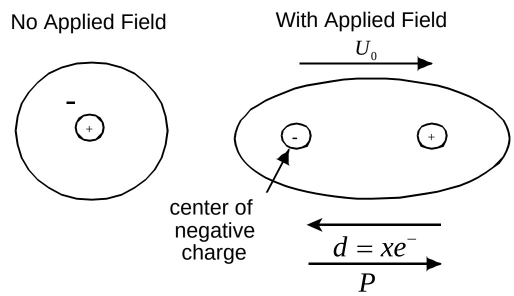
Maxwell's Equations

Diffraction and Interferometry

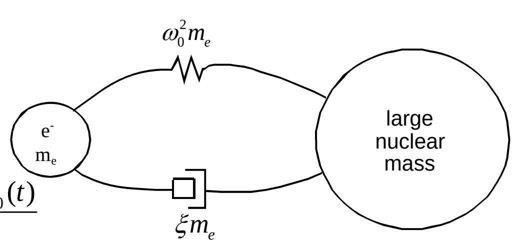
Dielectric Materials

(Fig. 2.1)

molecular charge distributions



harmonic oscillator model



$$\frac{\partial^2 x}{\partial t^2} + \xi \frac{\partial x}{\partial t} + \omega_0^2 x = \frac{-e^- U_0(t)}{m_e}$$

Properties of the Polarization

$$x(t) = \frac{e^{-}U_0(t)/m_e}{\left(\omega_0^2 - \omega^2\right) + j\omega\xi}$$

$$P(t) = \gamma e^{-} x(t) = \frac{\gamma\left(e^{-}\right)^2 U_0(t)/m_e}{\left(\omega_0^2 - \omega^2\right) + j\omega\xi} = \varepsilon_0 \chi U(t)$$

$U_0(t) =$ applied field
$U(t) =$ total field in material
$\gamma =$ molecular concentration
$\chi =$ dielectric susceptibility

Maxwell's Equations in Absorbing Media

Assume: source-free, infinite, homogeneous, isotropic, stationary, nonmagnetic, no free charge.

$$\nabla \times \boldsymbol{U}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} = -\mu_0 \frac{\partial \boldsymbol{H}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t} = \sigma \boldsymbol{U}(\boldsymbol{r},t) + \varepsilon_0 \frac{\partial \boldsymbol{U}(\boldsymbol{r},t)}{\partial t} + \frac{\partial \boldsymbol{P}(\boldsymbol{r},t)}{\partial t}$$

$$= \sigma \boldsymbol{U}(\boldsymbol{r},t) + \varepsilon_0 \left(1+\chi\right) \frac{\partial \boldsymbol{U}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \bullet \boldsymbol{D}(\boldsymbol{r},t) = 0$$

$$\nabla \bullet \boldsymbol{B}(\boldsymbol{r},t) = 0$$

$\mu_0 =$ permeability of free space
$\varepsilon_0 =$ permittivity of free space
$\sigma =$ conductivity

Maxwell's Equations in Absorbing Media

For harmonic fields of the form $\boldsymbol{U}(\boldsymbol{r},t) = \boldsymbol{A}(\boldsymbol{r})\exp(-j\omega t)$,

$$\begin{aligned}
\nabla \times \boldsymbol{H}(\boldsymbol{r},t) &= \sigma \boldsymbol{U}(\boldsymbol{r},t) + \varepsilon_0 \frac{\partial \boldsymbol{U}(\boldsymbol{r},t)}{\partial t} + \frac{\partial \boldsymbol{P}(\boldsymbol{r},t)}{\partial t} \\
&= \sigma \boldsymbol{U}(\boldsymbol{r},t) - j\omega\varepsilon_0 (\boldsymbol{1}+\chi)\boldsymbol{U}(\boldsymbol{r},t) \\
&= \left[\sigma - j\omega\varepsilon_0(\boldsymbol{1}+\chi)\right]\boldsymbol{U}(\boldsymbol{r},t) \\
&= -j\omega\varepsilon_0\left[(\boldsymbol{1}+\chi) - \frac{j\sigma}{\omega\varepsilon_0}\right]\boldsymbol{U}(\boldsymbol{r},t) \\
&= -j\omega\varepsilon_0\varepsilon_r \boldsymbol{U}(\boldsymbol{r},t) \quad .
\end{aligned}$$

$\varepsilon_r =$ relative permittivity (relative dielectric constant)

The Wave Equation in Absorbing Media

Application of the identity $\nabla \times \nabla \times \boldsymbol{U} = \nabla(\nabla \bullet \boldsymbol{U}) - \nabla^2 \boldsymbol{U}$ yields

$$\nabla(\nabla \bullet \boldsymbol{U}(t)) - \nabla^2 \boldsymbol{U}(t) = -\mu_0\varepsilon_0\varepsilon_r \frac{\partial^2 \boldsymbol{U}(t)}{\partial \boldsymbol{t}^2} = \mu_0\varepsilon_0\varepsilon_r\omega^2 \boldsymbol{U}(t)$$

$$-\nabla^2 \boldsymbol{U}(t) = \mu_0\varepsilon_0\varepsilon_r\omega^2 \boldsymbol{U}(t) \qquad \text{, or}$$

$$\nabla^2 \boldsymbol{U}(t) + \mu_0\varepsilon_0\varepsilon_r\omega^2 \boldsymbol{U}(t) = 0. \qquad \text{(HELMHOTLTZ EQUATION)}$$

For harmonic fields of the form $\boldsymbol{U}(\boldsymbol{r},t) = \boldsymbol{A}_0 \exp(j\boldsymbol{k}\bullet\boldsymbol{r})\exp(-j\omega t)$,

where $\boldsymbol{k} = k\hat{\boldsymbol{k}}$,

$$k = \frac{\omega}{c}\sqrt{\varepsilon_r} = \frac{\omega}{c}\boldsymbol{N} = k_r + jk_i \quad ,$$

and $N = \sqrt{\varepsilon_r}$ is the complex index of refraction.

Power Flow and the Poynting Vector

$$[U] = \mathrm{Vm^{-1}}$$

$$[H] = \mathrm{Am^{-1}}$$

$$[\boldsymbol{S} = \boldsymbol{U} \times \boldsymbol{H}] = \mathrm{VAm^{-2}} = \mathrm{Wm^{-2}}$$

IRRADIANCE:

$$\boldsymbol{U}(\boldsymbol{r},t) = \boldsymbol{A_0} \exp(j\boldsymbol{k}\bullet\boldsymbol{r})\exp(-j\omega t) \quad \text{and} \quad n = \mathrm{Re}(N),$$

$$I = |<\boldsymbol{S}>| = \frac{1}{2}c\varepsilon_0 n|\boldsymbol{A_0}|^2 \quad \mathrm{Wm^{-2}} \quad \text{in direction } \boldsymbol{k}.$$

Beer's Law

A solution for complex $N = \sqrt{\varepsilon_r} = n + \kappa j$ is

$$\boldsymbol{U}(\boldsymbol{r},t) = \boldsymbol{A_0} \exp\left(-\frac{\alpha}{2}\hat{\boldsymbol{k}}\bullet\boldsymbol{r}\right)\exp(-j\omega t) \quad ,$$

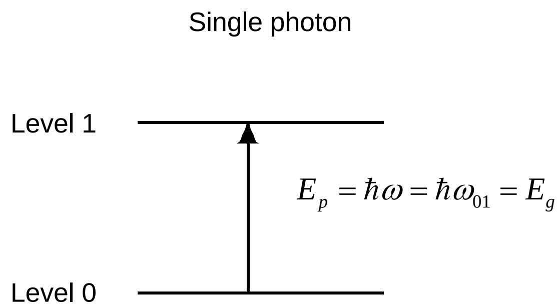
where $k_r = 2\pi n/\lambda$ and $\alpha = 2k_i = 4\pi\kappa/\lambda$ is the absorption coefficient.

$$I(z) = \frac{1}{2}c\varepsilon_0 n A_0^2 \exp(-\alpha z) \qquad \mathrm{Wm^{-2}}.$$

$$\hat{k} = \hat{z}$$

(Fig. 2.2)

Beer's Law via the Band Model

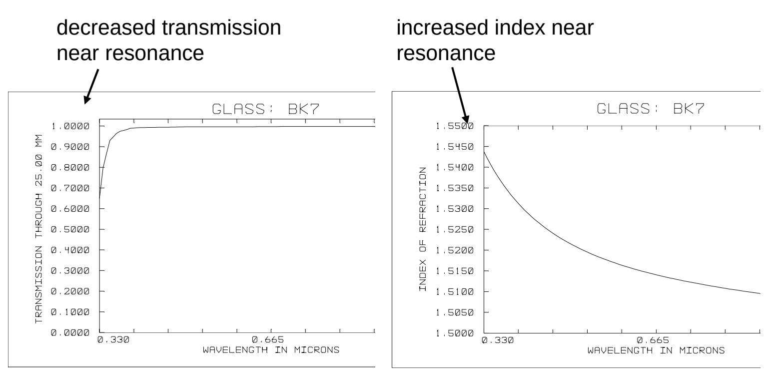


The band model is used to more accurately describe the absorption process, but a similar frequency dependence is found.

(Fig. 2.3)

Frequency Dependence of *N*

Either the harmonic oscillator or the band model shows that n and α depend on temporal frequency.



Wave Equation in Free Space

Maxwell's equations in free space:

$$\nabla \times \boldsymbol{U}(\boldsymbol{r},t) = -\mu_0 \frac{\partial \boldsymbol{H}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \varepsilon_0 \frac{\partial \boldsymbol{U}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \bullet \boldsymbol{D}(\boldsymbol{r},t) = 0$$

$$\nabla \bullet \boldsymbol{B}(\boldsymbol{r},t) = 0$$

Lead to the free-space wave equation:

$$\nabla^2 \boldsymbol{U}(\boldsymbol{r},t) - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{U}(\boldsymbol{r},t)}{\partial \boldsymbol{t}^2} = 0 \quad .$$

(Helmholtz Eq. in free space.)