Solutions to the Wave Equation

(Part B – Vector Waves and Polarization)

Diffraction and Interferometry

Plane Waves

Three-dimensional Helmholtz wave equation in free space:

$$\nabla^2 \boldsymbol{U} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{U}}{\partial t^2} = 0$$

Solution traveling in direction $\hat{k}$:

$$\boldsymbol{U}(\boldsymbol{r},t) = \boldsymbol{U}_0 e^{j(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = \|\boldsymbol{U}_0\| \hat{\boldsymbol{a}} e^{j(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t+\phi)} = A_0 \hat{\boldsymbol{a}} e^{j(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

where $\boldsymbol{r} = r\hat{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$

and $\boldsymbol{k} = k\hat{\boldsymbol{k}} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}} = k(\alpha\hat{\boldsymbol{x}} + \beta\hat{\boldsymbol{y}} + \gamma\hat{\boldsymbol{z}})$.

Note that $\boldsymbol{k}\cdot\boldsymbol{r} - \omega t = k_x x + k_y y + k_z z - \omega t = \text{constant}$

represents planes in space of constant phase. Hence, we call this solution a *plane wave*.

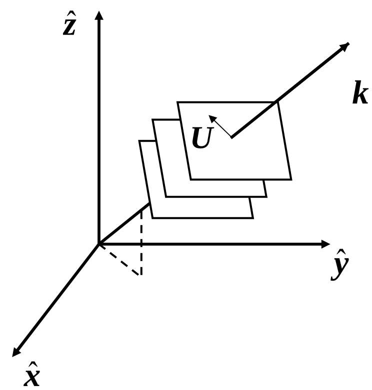
Plane Waves

$$\boldsymbol{U}(\boldsymbol{r},t) = \boldsymbol{U_0} e^{j(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)} = \|\boldsymbol{U_0}\| \hat{\boldsymbol{a}} e^{j(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t+\phi)} = A_0 \hat{\boldsymbol{a}} e^{j(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)}$$

Wave fronts $\perp \boldsymbol{k}$.

Max separation between planes = λ.

$\hat{\boldsymbol{a}} \perp \hat{\boldsymbol{k}}$ in linear, isotropic media



Vector Spherical Waves

In the wave zone, where $r \gg \frac{\lambda}{2\pi}$,

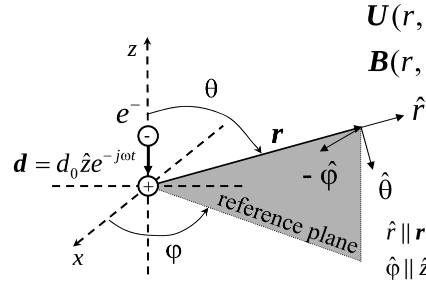
$\boldsymbol{U}(r,\theta,\varphi)$

$\boldsymbol{B}(r,\theta,\varphi)$

$U_r \approx 0$

$U_\theta \approx \frac{e d_0 k^2 \sin\theta}{4\pi\varepsilon_0 r} e^{j(kr-\omega t)}$

$B_\varphi \approx \frac{c\mu_0 d_0 k^2 \sin\theta}{4\pi r} e^{j(kr-\omega t)}$



$\hat{r} \parallel \boldsymbol{r}$

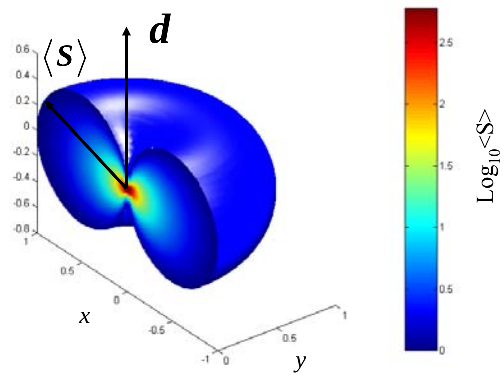
$\hat{\varphi} \parallel \hat{z} \times \hat{r}$

$\hat{\theta} \parallel \hat{\varphi} \times \hat{r}$

This configuration of time-varying EM fields, which is called *dipole radiation*, is very important in understanding the physical interpretation of wave optics.

Vector Spherical Waves
(Power Flow)

Although the wavefronts for a dipole radiator in the wave zone are the same as a scalar spherical wave, the EM radiation is amplitude modulated as a function of θ.



Scalar Spherical Waves

A solution to the wave equation in spherical coordinates is:

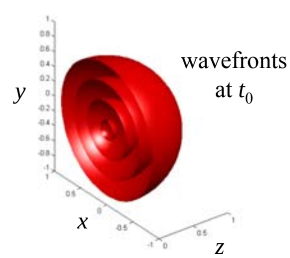
$$U(r,t) = \frac{1}{r} e^{j(kr - \omega t)} \quad ,$$

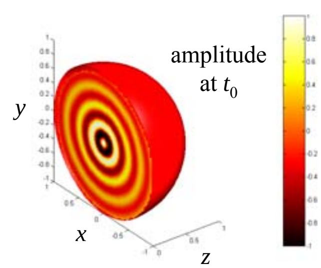
where

$$r = \sqrt{x^2 + y^2 + z^2}$$

and $\psi = kr - \omega t$.

Wave crests (also called *wavefronts*) at ψ = constant expand outwardly from the source point as a function of time. Wavefronts are perpendicular to ***r***.





Linearity

Since the electric field is a vector quantity, a vector summation must be performed to find the electric field resulting from the summation of several electric fields.

$$\boldsymbol{U}_{total} = \boldsymbol{U}_1 + \boldsymbol{U}_2 + \boldsymbol{U}_3 + \cdots$$

This linear superposition is only approximately true in the presence of matter. Deviations from linearity are observed at high intensities produced by lasers when the electric fields approach the electric fields comparable to atomic fields (non-linear optics). In these notes we will consider only situations where linear superposition is valid.

Polarization

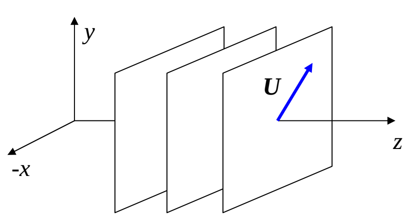
Consider the case of an EM plane wave traveling in the +*z* direction.

E oscillates perpendicular to *z*.

General solution:

$$\boldsymbol{U}(z,t) = \boldsymbol{U}_0 e^{j(kz-\omega t)} = \left[A_x \hat{\boldsymbol{x}} + A_y e^{j\phi} \hat{\boldsymbol{y}} \right] e^{j(kz-\omega t)} \quad A_x, A_y \in \mathrm{Re}\{\ \}$$

We will now trace the tip of the electric vector for several special cases.



We will look at both:

fixed $z = z_0$ with t = variable

fixed $t = t_0$ with z = variable

Polarization – Linear

$\phi = 0$

Special case where $A_x = A_y = A_0$:

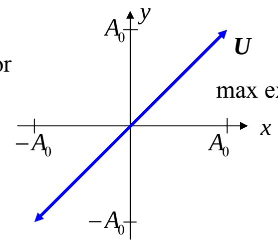
$$\boldsymbol{U}(z,t) = \boldsymbol{U}_0 e^{j(kz-\omega t)} = \sqrt{2}A_0 \hat{\boldsymbol{a}} e^{j(kz-\omega t)} = A_0\left(\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\right)e^{j(kz-\omega t)}$$

$$\mathrm{Re}\left[\boldsymbol{U}(z,t)\right] = A_0\left(\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\right)\cos\left(kz-\omega t\right)$$ ← This is the physical wave.

Consider $z = z_0$, where $\mathrm{Re}\left[\boldsymbol{U}(z_0,t)\right] = A_0\left(\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\right)\cos\left(kz_0-\omega t\right)$

Trace the tip of the electric vector in the (x,y) plane in time:

This is true for *all* z_0.



max extension = $\pm\sqrt{2}A_0$

$$\hat{\boldsymbol{a}} = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\right)$$

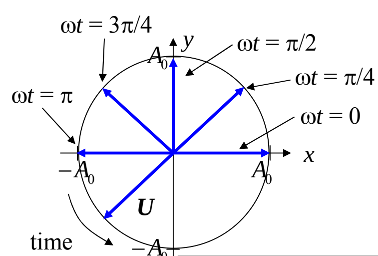
Polarization – Left Circular

$A_x = A_y = A_0$
$\phi = \pi/2$

$$\boldsymbol{U}(z,t) = A_0\left[\hat{\boldsymbol{x}} + e^{j\pi/2}\hat{\boldsymbol{y}}\right]e^{j(kz-\omega t)} \qquad \hat{\boldsymbol{a}} = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{x}} + e^{j\pi/2}\hat{\boldsymbol{y}}\right)$$

Consider $z = 0$, where

$$\mathrm{Re}\left\{\boldsymbol{U}(0,t)\right\} = A_0\left[\cos(\omega t)\hat{\boldsymbol{x}} + \cos(\pi/2-\omega t)\hat{\boldsymbol{y}}\right] = A_0\left[\cos(\omega t)\hat{\boldsymbol{x}} + \sin(\omega t)\hat{\boldsymbol{y}}\right]$$

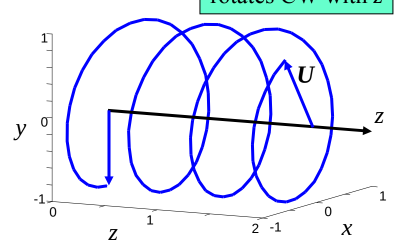


rotates CCW with time

Consider $t = 0$, where

$$\mathrm{Re}\left\{\boldsymbol{U}(z,0)\right\} = A_0\left[\cos(kz)\hat{\boldsymbol{x}} + \cos(\pi/2+kz)\hat{\boldsymbol{y}}\right] = A_0\left[\cos(kz)\hat{\boldsymbol{x}} - \sin(kz)\hat{\boldsymbol{y}}\right]$$

rotates CW with z



Polarization – Right Circular

$A_x = A_y = A_0$
$\phi = -\pi/2$

$$U(z,t) = A_0\left[\hat{x} + e^{-j\pi/2}\hat{y}\right]e^{j(kz-\omega t)} \qquad \hat{a} = \frac{1}{\sqrt{2}}\left(\hat{x} + e^{-j\pi/2}\hat{y}\right)$$

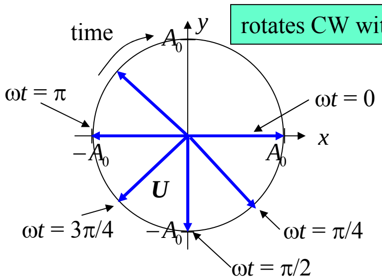
Consider $z = 0$, where

$$\mathrm{Re}\{U(0,t)\} = A_0\left[\cos(\omega t)\hat{x} + \cos(-\pi/2 - \omega t)\hat{y}\right] = A_0\left[\cos(\omega t)\hat{x} - \sin(\omega t)\hat{y}\right]$$

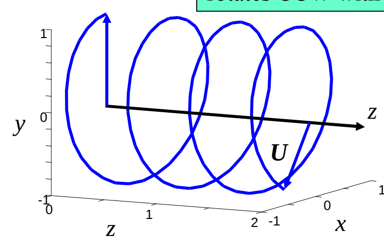
Consider $t = 0$, where

$$\mathrm{Re}\{U(z,0)\} = A_0\left[\cos(kz)\hat{x} + \cos(-\pi/2 + kz)\hat{y}\right] = A_0\left[\cos(kz)\hat{x} + \sin(kz)\hat{y}\right]$$

rotates CW with time



rotates CCW with z



Polarization – Elliptical

A_x, A_y
ϕ_0

$$U(z,t) = \left[A_x\hat{x} + A_y e^{j\phi}\hat{y}\right]e^{j(kz-\omega t)}$$, where

$$U_x = A_x\cos\left(kz - \omega t\right) \qquad (1)$$

$$U_y = A_y\cos\left(kz - \omega t + \phi\right) \qquad (2)$$

Combination of (1) and (2) gives:

$$\left(\frac{U_x}{A_x}\right)^2 + \left(\frac{U_y}{A_y}\right)^2 - 2\left(\frac{U_x}{A_x}\right)\left(\frac{U_y}{A_y}\right)\cos\phi = \sin^2\phi$$,

which is an equation of an ellipse.

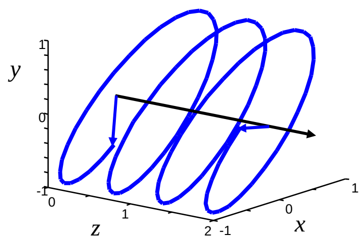
In other words,

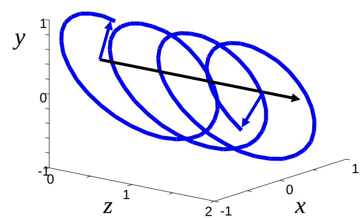
The tip of the electric vector specified by U_x and U_y trace an ellipse in any plane z that is determined by A_x, A_y and ϕ.

Polarization – Elliptical

$A_x = A_y = 1$
$\phi = \pi / 4$

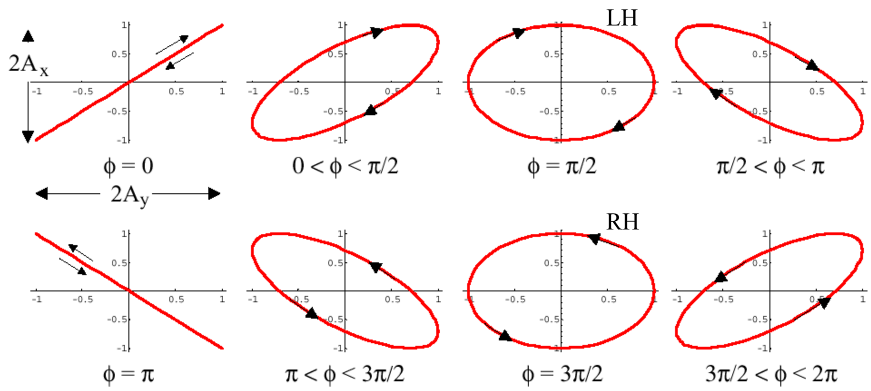
$A_x = A_y = 1$
$\phi = 5\pi / 4$





Polarization – Elliptical

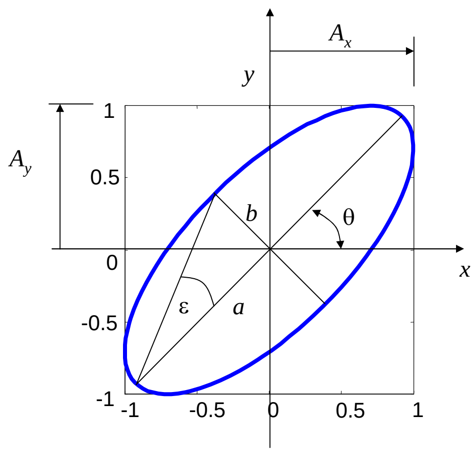
(Arrows show trace in *z* direction)



This slide from Jim Wyant, 2000.

Polarization – Ellipticity

Look into wave and observe the trace at z_0 as a function of time



$$\tan\alpha = \frac{A_y}{A_x} \quad (0 \le \alpha \le \pi/2)$$

$$\tan\varepsilon = \mp\frac{b}{a} \quad (-\pi/4 \le \varepsilon < \pi/4)$$

$$\tan 2\theta = (\tan 2\alpha)\cos\phi \quad (0 \le \theta < \pi)$$

$$\sin 2\varepsilon = (\sin 2\alpha)\sin\phi$$

ε is often called the *ellipticity*

- ε is RHC

+ ε is LHC

Polarization – Jones* Calculus

Look into wave and observe the trace at z_0 as a function of time

The values of constants A_x, A_y and ϕ determine the *state of polarization* of the plane wave. Various states include *linear*, *circular* and *elliptical*.

We can write the wave conveniently with these constants in a column vector:

$$\boldsymbol{U}(z,t) = \begin{pmatrix} A_x \\ A_y e^{j\phi} \end{pmatrix} e^{j(kz-\omega t)} \quad .$$

As the wave propagates, the state of polarization does not change unless the wave encounters and optical element. The *Jones vector* is the state of polarization written as as column vector.

$$\text{Jones vector } = J = \begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} A_x \\ A_y e^{j\phi} \end{pmatrix}$$

*R. Clark Jones, *JOSA A*, 31, 488-493 (1941), provided on web site. Actually, there are a series of eight papers.

Polarization – Jones Vector Examples

Common constants may be divided out of each Jones vector component.

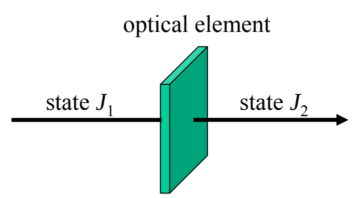
Example: $J = \begin{pmatrix} 3e^{j\pi/2} \\ 9e^{j\pi/2} \end{pmatrix} = 3e^{j\pi/2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Various states of polarization:

linear x	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	linear y	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
linear +45°	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	linear -45°	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
RHC	$\begin{pmatrix} 1 \\ -j \end{pmatrix}$	LHC	$\begin{pmatrix} 1 \\ j \end{pmatrix}$

Polarization – Jones Matrices for Optical Elements

Optical elements can affect the state of polarization as a plane wave passes through them.

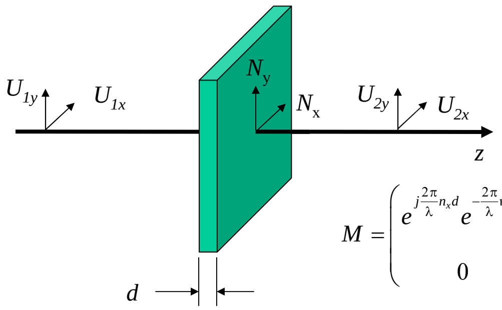


If we assume that the element acts linearly, the change of state through an element can be represented by a 2-by-2 matrix *M*.

$$J_2 = MJ_1 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} J_1 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} J_{1x} \\ J_{1y} \end{pmatrix} = \begin{pmatrix} m_{11}J_{1x} + m_{12}J_{1y} \\ m_{21}J_{1x} + m_{22}J_{1y} \end{pmatrix}$$

Polarization – Jones Matrices for Optical Elements

Consider optical elements with polarization-dependent refractive index.



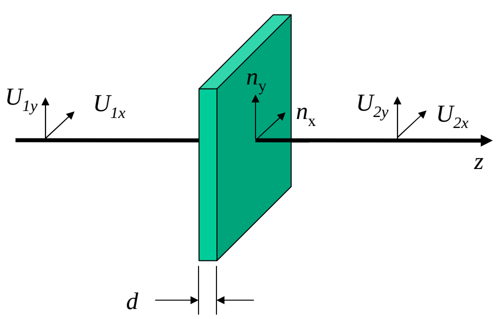
$$N_x = n_x + j\kappa_x$$
$$N_y = n_y + j\kappa_y$$

$$M = \begin{pmatrix} e^{j\frac{2\pi}{\lambda}n_x d} e^{-\frac{2\pi}{\lambda}\kappa_x d} & 0 \\ 0 & e^{j\frac{2\pi}{\lambda}n_y d} e^{-\frac{2\pi}{\lambda}\kappa_y d} \end{pmatrix}$$

Retardation plate: $n_x \neq n_y,\ \kappa_x \approx \kappa_y \approx 0$
Polarizer plate: $n_x \approx n_y,\ \kappa_x \neq \kappa_y$

Polarization – Jones Matrix Examples

Uniaxial Crystals



$n_x = n_o$
$n_y = n_e$ (OA)
$n_z = n_o$

Uniaxial crystals have two primary refractive indices, depending on how the light beam is polarized. n_o (for *ordinary*) and n_e (for *extraordinary*). A common configuration is shown above. n_e defines the *optical axis* OA. A *positive* crystal, like quartz, has $n_e > n_o$. A *negative* crystal, like calcite, has $n_o > n_e$.

Polarization – Jones Matrix Examples

Quarter-Wave Plate

A quartz quarter-wave plate is a uniaxial crystal with

$n_0 = n_x = 1.544$
$n_e = n_y = 1.553$
$\kappa_e = \kappa_o \approx 0$
at $\lambda = 500$ nm

The thickness d is fabricated so that the phase difference between x and y polarization components through the plate is $\pi/2$.

$$J_2 = c\begin{pmatrix} 1 & 0 \\ 0 & e^{j\pi/2} \end{pmatrix} J_1 = M_{QWP} J_1 = \begin{pmatrix} e^{j\frac{2\pi}{\lambda}n_o d} & 0 \\ 0 & e^{j\frac{2\pi}{\lambda}n_e d} \end{pmatrix} J_1 \quad ,$$

where c is a constant. So,

$$c\begin{pmatrix} 1 & 0 \\ 0 & e^{j\pi/2} \end{pmatrix} = \begin{pmatrix} e^{j\frac{2\pi}{\lambda}n_o d} & 0 \\ 0 & e^{j\frac{2\pi}{\lambda}n_e d} \end{pmatrix} = e^{j\frac{2\pi}{\lambda}n_o d}\begin{pmatrix} 1 & 0 \\ 0 & e^{j\frac{2\pi}{\lambda}(n_e - n_o)d} \end{pmatrix} \quad .$$

Polarization – Jones Matrix Examples

Quarter-Wave Plate

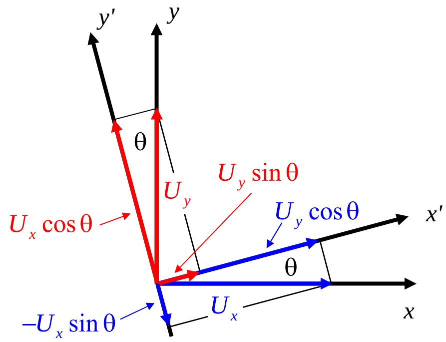
Comparison of terms yields

$$\frac{\pi}{2} = \frac{2\pi}{\lambda}(n_e - n_o)d, \text{ or}$$

$$d = \frac{\lambda}{4}\frac{1}{n_e - n_o} = \frac{500 \text{ nm}}{4(1.553\text{-}1.544)} = 14 \times 10^3 \text{ nm} = 14\mu\text{m}$$

Polarization – Rotated Elements

In order to describe the effect of a rotated optical element, we can project the (x,y) electric fields onto the element with the *rotation matrix*, M_R.



$$U'_x = U_x \cos\theta + U_y \sin\theta$$
$$U'_y = -U_x \sin\theta + U_y \cos\theta$$

$$\begin{pmatrix} U'_x \\ U'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} U_x \\ U_y \end{pmatrix}$$

$$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Polarization – Jones Matrix Examples

Rotated Quarter-Wave Plate

Consider rotating a quarter-wave plate by 45°. Follow these steps to find the effect on transmitted light:

1.) Decompose the incident light into components. Application of the rotation matrix produces a description of the light field *in the coordinate frame of the rotated plate.*

$$J'_1 = M_\theta J_1$$

2.) Apply the quarter-wave plate matrix.

$$J'_2 = M_{QWP} J'_1 = M_{QWP} M_\theta J_1$$

3.) Rotate back into the (x,y) coordinate system.

$$J_2 = M_\theta^\dagger J'_2 = M_\theta^\dagger M_{QWP} J'_1 = M_\theta^\dagger M_{QWP} M_\theta J_1 = M_{QWP}^\theta J_1$$

Notice that

$$M_{QWP}^\theta = M_\theta^\dagger M_{QWP} M_\theta \qquad (\dagger = \text{transpose})$$

Polarization – Jones Matrix Examples

Rotated Quarter-Wave Plate (cont'd)

$$M_{QWP}^{45^\circ} = M_{45^\circ}^{\dagger} M_{QWP} M_{45^\circ}$$

$$= \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix} \begin{pmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -j & j \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1+j & 1-j \\ 1-j & 1+j \end{pmatrix} = \frac{1+i}{2} \begin{pmatrix} 1 & -j \\ -j & 1 \end{pmatrix}$$

A quarter-wave plate rotated at 45º has a very simple equivalent matrix.

Polarization – Jones Matrix Examples

Rotated Quarter-Wave Plate (cont'd)

Look at the effect of a rotated QWP on *x*-polarized light.

$$J_2 = M_{QWP}^{45^\circ} J_1$$

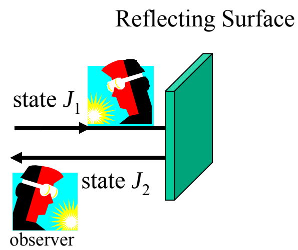
$$= \frac{1+j}{2} \begin{pmatrix} 1 & -j \\ -j & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1+j}{2} \begin{pmatrix} 1 \\ -j \end{pmatrix}$$

The effect of a QWP (rotated at 45º) on *x*-polarized light is to change the state of polarization to RHC.

Polarization – Jones Matrix Examples

Reflection

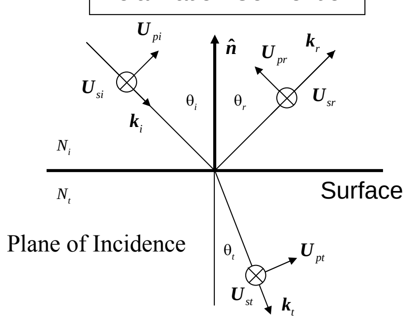


Remember, we look *into* the wave to determine the state of polarization. To be consistent, we define *p* and *s* polarization states relative to the surface normal and the incident *k* vector. The representation of a reflective surface is:

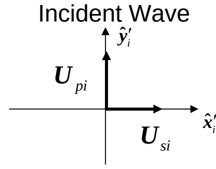
$$J_2 = M_r J_1 = \begin{pmatrix} r_s & 0 \\ 0 & r_p \end{pmatrix} J_1$$

Polarization – Reflection

Polarization Convention

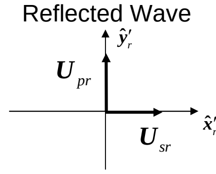


Incident Wave



Right-Handed Coordinates

Reflected Wave



Reflected System Matrix

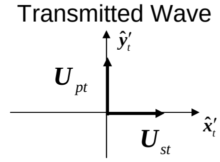
$$\boldsymbol{M}_r = \begin{pmatrix} r_s & 0 \\ 0 & r_p \end{pmatrix}$$

Fresnel's Equations

$$r_s = \frac{N_i \cos\theta_i - N_t \cos\theta_t}{N_i \cos\theta_i + N_t \cos\theta_t} \qquad r_p = \frac{N_t \cos\theta_i - N_i \cos\theta_t}{N_t \cos\theta_i + N_i \cos\theta_t}$$

$$t_s = \frac{2 N_i \cos\theta_i}{N_i \cos\theta_i + N_t \cos\theta_t} \qquad t_p = \frac{2 N_i \cos\theta_i}{N_t \cos\theta_i + N_i \cos\theta_t}$$

Transmitted Wave



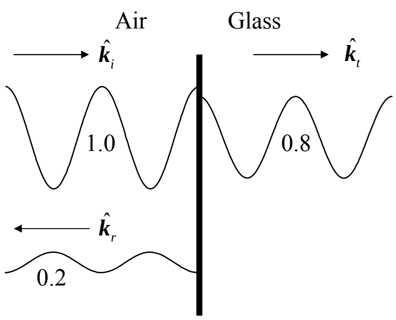
Transmitted System Matrix

$$\boldsymbol{M}_t = \begin{pmatrix} t_s & 0 \\ 0 & t_p \end{pmatrix}$$

Polarization – Reflection

$\theta_i = 0$

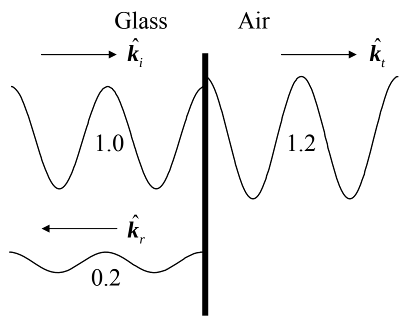
Air/Glass Interface



$N_i = 1$ $N_t = 1.5$

$$\boldsymbol{M}_r = \begin{pmatrix} -0.2 & 0 \\ 0 & 0.2 \end{pmatrix}$$

Glass/Air Interface

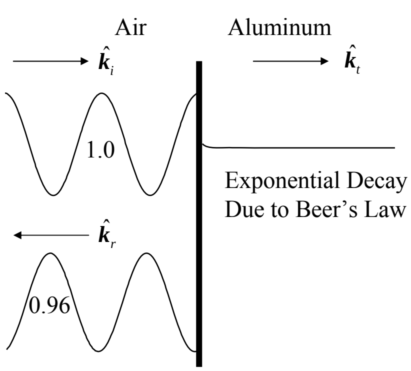


$N_i = 1.5$ $N_t = 1$

$$\boldsymbol{M}_r = \begin{pmatrix} 0.2 & 0 \\ 0 & -0.2 \end{pmatrix}$$

Polarization – Reflection

$\theta_i = 0$



$N_i = 1$ $N_t = 1.02 + 6.85j$

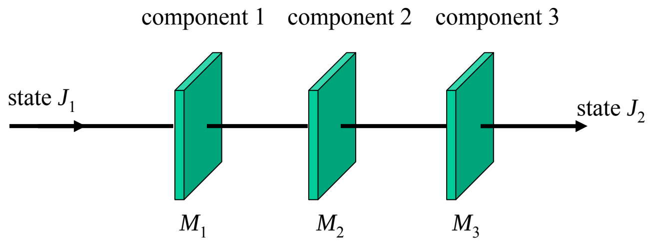
(for $\lambda = 563$nm)

$$\boldsymbol{M}_r = \begin{pmatrix} -0.96e^{j2\pi 0.045} & 0 \\ 0 & 0.96e^{j2\pi 0.045} \end{pmatrix}$$

$$= 0.96e^{j2\pi 0.045} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Polarization – Jones Matrix Systems

Cascaded Components

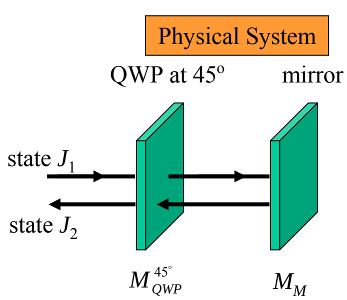


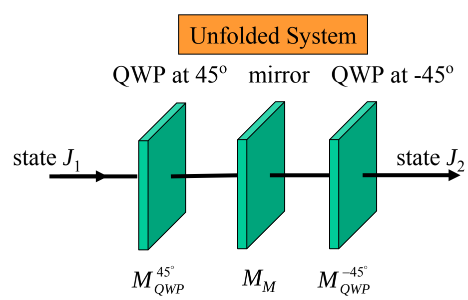
$$J_2 = M_3 M_2 M_1 J_1 = M_{SYS} J_1$$

Cascaded elements can be represented by algebraically multiplying component matrices into a system matrix M_{SYS}.

Polarization – Jones Matrix Examples

Combination of a Quarter-Wave Plate and a Mirror





$$J_2 = M_{QWP}^{-45^\circ} M_M M_{QWP}^{45^\circ} J_1$$

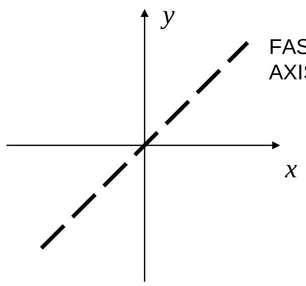
$$M_M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad M_{QWP}^{45^\circ} = \frac{1+j}{2}\begin{pmatrix} 1 & -j \\ -j & 1 \end{pmatrix}$$

$$M_{QWP}^{-45^\circ} = \frac{1+j}{2}\begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix}$$

Polarization – Jones Matrix Examples

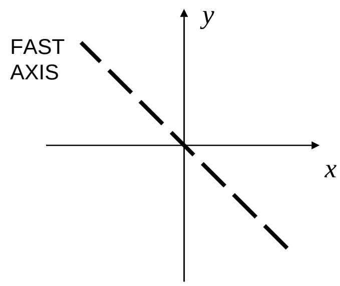
Quarter-Wave Plate In Reflection

Before Mirror



$$M_{QWP}^{45^\circ} = \frac{1+j}{2}\begin{pmatrix} 1 & -j \\ -j & 1 \end{pmatrix}$$

After Mirror



$$M_{QWP}^{-45^\circ} = \frac{1+j}{2}\begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix}$$

Polarization – Jones Matrix Examples

Combination of a Quarter-Wave Plate and a Mirror (cont'd)

$$J_2 = \frac{1+j}{2}\begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\frac{1+j}{2}\begin{pmatrix} 1 & -j \\ -j & 1 \end{pmatrix}J_1$$

$$= \frac{j}{2}\begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -j \\ -j & 1 \end{pmatrix}J_1$$

$$= \frac{j}{2}\begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix}\begin{pmatrix} -1 & j \\ -j & 1 \end{pmatrix}J_1$$

$$= \frac{j}{2}\begin{pmatrix} 0 & 2j \\ -2j & 0 \end{pmatrix}J_1$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}J_1$$

Notice the effect on *x*-polarized light:

If $J_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The state of polarization is now *y* polarized.

Polarization – Jones Matrix Examples

Linear polarizer x	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Linear polarizer y	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
HWP	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
General retarder	$e^{j\psi}\begin{pmatrix} 1 & 0 \\ 0 & e^{j\phi} \end{pmatrix}$