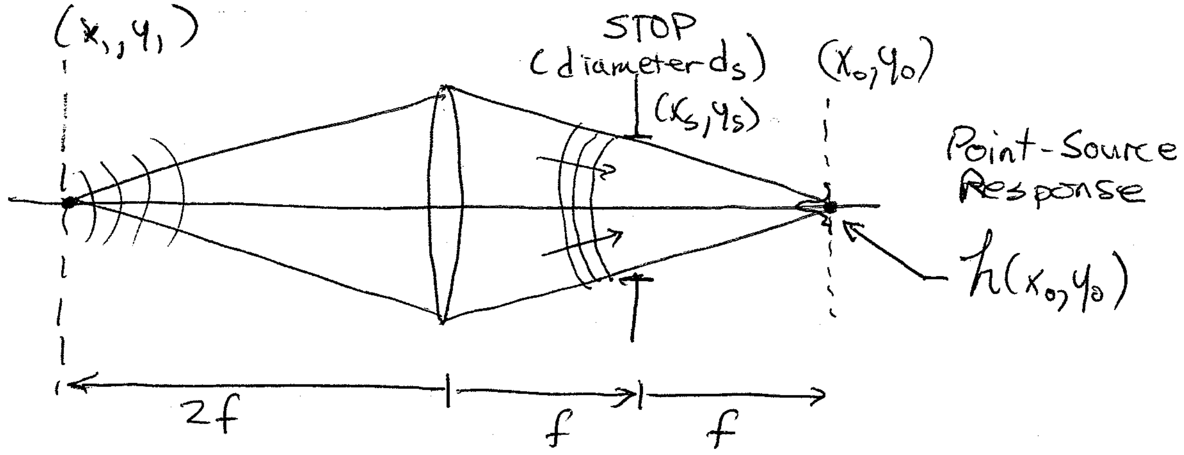
Coherent Imaging:

Simple optical system with point-source object:



on-axis point source

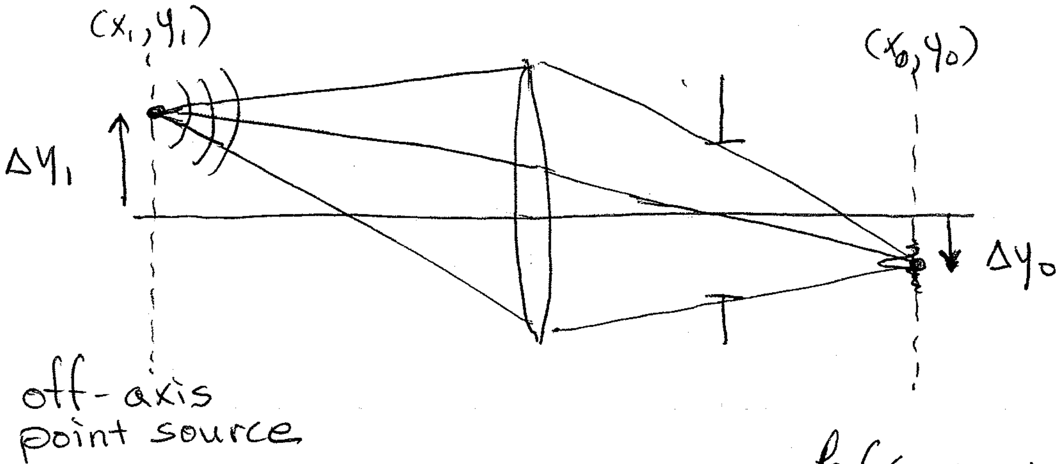
$$h(x_0,y_0) = \overset{\text{complex}}{K}\, F_{\eta=\frac{y_0}{\lambda f}}\, F_{\xi=\frac{x_0}{\lambda f}}\left[\overset{\text{Transmittance of stop.}}{T(x_s,y_s)}\right]$$

The on-axis point source is analogous to the unit pulse in electronics, which contains an infinite range of temporal frequencies. Here, we know from Weyl's integral that the point source contains an infinite range of plane-wave components. Each plane-wave component corresponds to a spatial frequency in the object plane.

$$\frac{e^{j(kr-\omega t)}}{r} = -\frac{j}{\lambda}\int_{-\infty}^{\infty} e^{j(\vec{k}\cdot\vec{r}-2\pi\nu t)}\,\underset{\leftarrow\text{SOLID ANGLE}}{d\omega}$$

If the system is linear, changes in source amplitude result in proportional changes in h.

Shift Invariance:



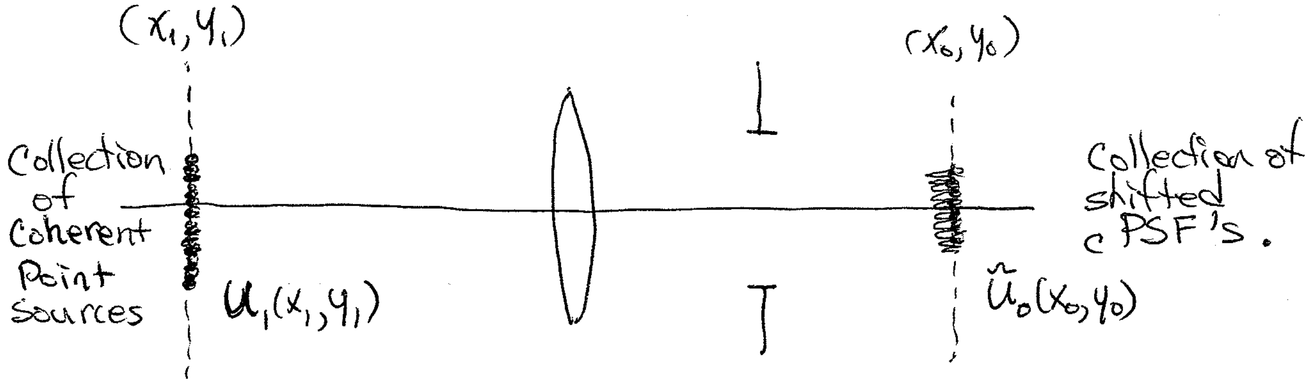
$$m_T = \frac{\Delta y_0}{\Delta y_1}$$

$h(x_0, y_0 - \Delta y_0)$

Same as $h(x_0, y_0)$, but displaced by geometric magnification.

This relationship must be true for all Δy_1 in the object field. Field-dependent aberrations cause a problem here.

$h(x_0, y_0)$ is the <u>Coherent Point Spread Function</u> cPSF for linear and shift-invariant systems.



Consider collection of point sources in $x_1 y_1$, each producing a shifted cPSF. Exactly, object is a collection of Huygens wavelet radiators, but, several λ's away, essentially spherical waves. Can write:

$$\boxed{\tilde{U}_0(x_0,y_0) = \frac{1}{m_T^2} U_1\left(\frac{x_0}{m_T}, \frac{y_0}{m_T}\right) ** h(x_0,y_0)}$$

↑ Ideal geometrical image

↑ Image w/o leading quadratic phase and axial phase.

Frequency analysis:

$$F_\eta F_\xi\left[\tilde{U}_0(x_0,y_0)\right] = \frac{1}{m_T^2} F_\eta F_\xi\left[U_1\left(\frac{x_0}{m_T}, \frac{y_0}{m_T}\right)\right] F_\eta F_\xi\left[h(x_0,y_0)\right]$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$O_0(\xi,\eta) = O_1(m_T\xi, m_T\eta)\, H(\xi,\eta)$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

$$\begin{pmatrix}\text{Angular spectrum}\\ \text{of image}\end{pmatrix} = \begin{pmatrix}\text{Scaled Angular}\\ \text{spectrum of}\\ \text{object}\end{pmatrix} \begin{pmatrix}\underline{\text{Coherent}}\\ \underline{\text{Transfer}}\ \underline{\text{Function}}\\ \text{(CTF)}\end{pmatrix}$$

What is $H(\xi,\eta)$?

$$H(\xi,\eta) = F_{\eta} F_{\xi}\left[h(x_0,y_0)\right]$$

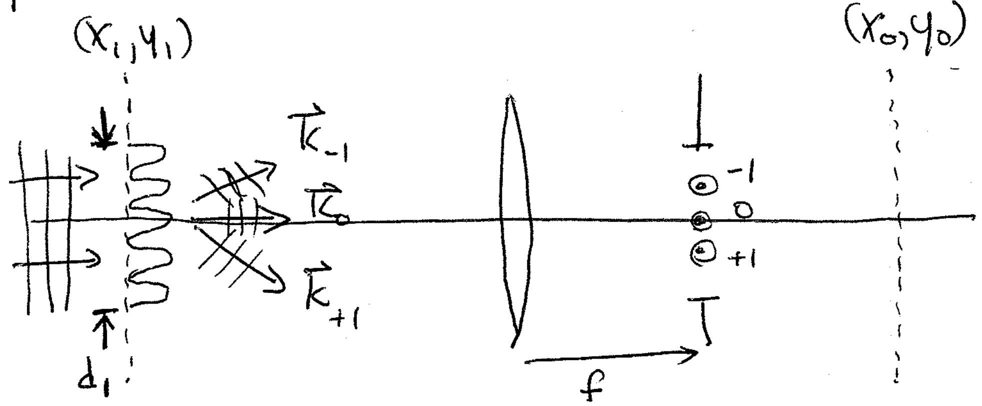
$$= F_{\eta} F_{\xi}\left\{ F_{\frac{y_0}{\lambda f}} F_{\frac{x_0}{\lambda f}}\left[T(x_s,y_s)\right]\right\}$$

$$= T(-\xi\lambda f, -\eta\lambda f)$$

$\therefore H(\xi,\eta)$ is simply a scaled and reflected pupil function!

Cosine Object:

Simple cosine object. On-axis pw illumination.



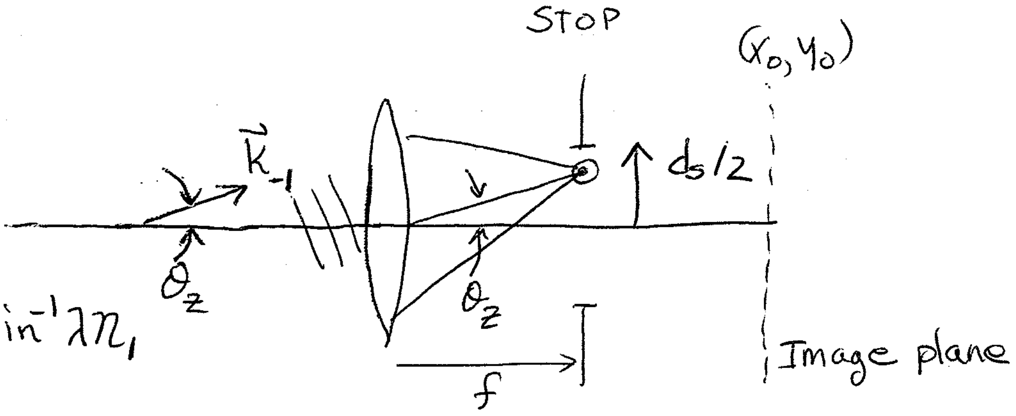
$$u_1(x_1,y_1) = \text{circ}\left(\frac{\rho_1}{d_1}\right)\left[\frac{1}{2}+\frac{1}{2}\cos(2\pi\eta_1 y_1)\right]$$

$$A_{z=0}(\xi,\eta) = O_1(\xi,\eta)$$

$$= \frac{\pi d_1^2}{4}\,\text{somb}\left(d_1\sqrt{\xi^2+\eta^2}\right) ** \left[\overset{\text{0 order}}{\overset{\downarrow}{\frac{1}{2}\delta(\eta)}} + \overset{\text{-1 order}}{\overset{\downarrow}{\frac{1}{4}\delta(\eta-\eta_1)}} + \overset{\text{+1 order}}{\overset{\downarrow}{\frac{1}{4}\delta(\eta+\eta_1)}}\right]\delta(\xi)$$

Three plane wave components in the object's angular spectrum are focused at f behind the lens.

At the lens:



$\theta_z = \sin^{-1} \lambda \eta_1$

Each plane-wave component focuses to a point in the aperture stop. Therefore, the distribution in the aperture stop is the object's spatial frequency distribution.

Higher $\eta_1 \Rightarrow$ Larger θ_z

Maximum spatial frequency passed by system:

$$(\theta_z)_{max} = \tan^{-1}\left(\frac{d_s/2}{f}\right)$$

$$\beta_{max} = \sin(\theta_z)_{max} \approx \frac{d_s}{2f} = NA$$

$$\eta_{max} = \frac{\beta_{max}}{\lambda} = \boxed{\frac{NA}{\lambda}}$$

Note: $\sin(\theta_z)_{max}$ = sine of marginal ray angle.

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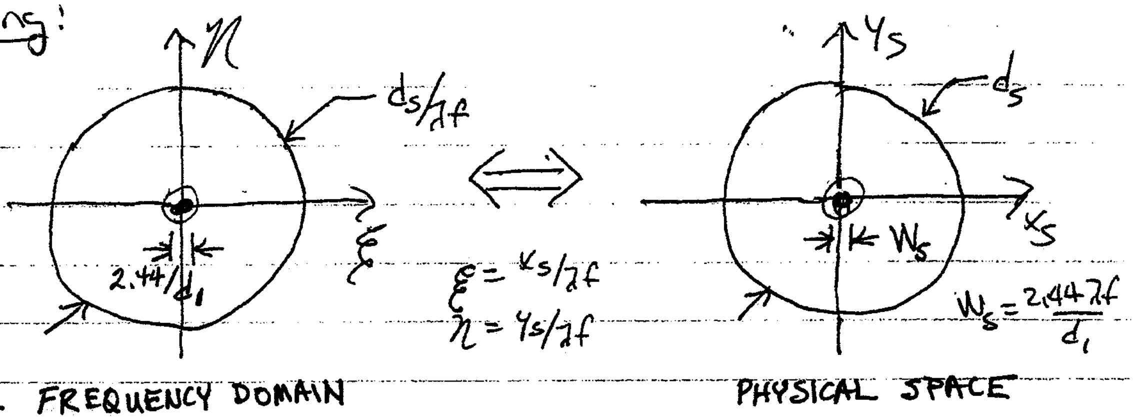
PHYSICAL INTERPRETATION OF $H(\xi,\eta)$:

NOTE THAT $H(\xi,\eta)$ IS THE TRANSMISSION THROUGH THE STOP. THAT IS, THE FIELD IN THE STOP IS A REPRESENTATION OF THE FREQUENCY SPECTRUM OF THE IMAGE.

No Grating:

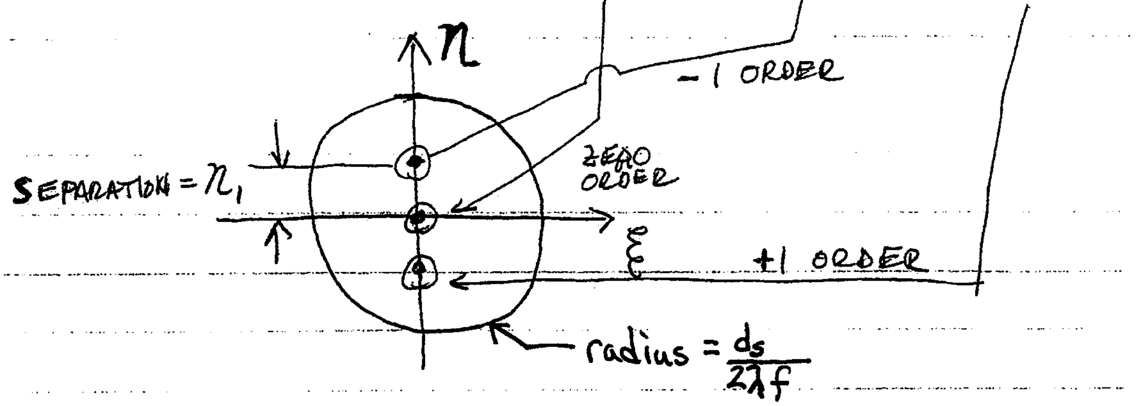


$\xi = x_s/\lambda f$
$\eta = y_s/\lambda f$

With Grating: $(m_T = -1)$

$$O_1(\xi,\eta) = \frac{1}{2} \mathcal{F}_\eta \mathcal{F}_\xi \left\{ \operatorname{circ}\left(\frac{\sqrt{x_1^2+y_1^2}}{d_1}\right) \left[1+\cos(2\pi\eta_1 y)\right] \right\}$$

$$= \frac{\pi}{4} d_1^2 \operatorname{somb}\left(d_1\sqrt{\xi^2+\eta^2}\right) ** \left[\frac{1}{2}\delta(\xi,\eta) + \frac{1}{4}\delta(\xi,\eta-\eta_1) + \frac{1}{4}\delta(\xi,\eta+\eta_1)\right]$$



PHYSICAL INTERPRETATION OF $H(\xi,\eta)$: (con't)

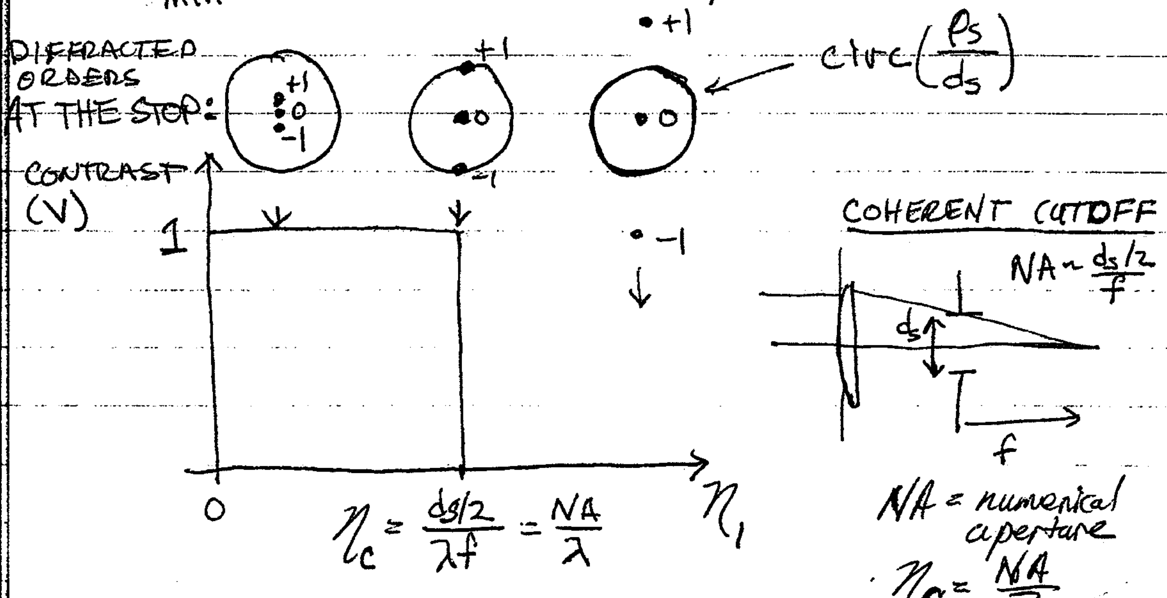
IN THE IMAGE PLANE, IF $\eta_1 < \frac{d_s/2}{\lambda f}$ AND $W_s << d_s$,
$(m_T = 1)$

$$|\tilde{U}_o(x_o,y_o)| \sim \frac{1}{2}\mathrm{circ}\left(\frac{\sqrt{x_o^2+y_o^2}}{d_1}\right)\left[1+\cos(2\pi\eta_1 y_o)\right]$$

$$I_o(x_o,y_o) = |U_o(x_o,y_o)|^2 \sim \mathrm{circ}^2\left(\frac{\sqrt{x_o^2+y_o^2}}{d_1}\right) \times\left[1+2\cos(2\pi\eta_1 y_o)+\cos^2(2\pi\eta_1 y_o)\right]$$

$$= \mathrm{circ}^2\left(\frac{\sqrt{x_o^2+y_o^2}}{d_1}\right)\left[3/2+2\cos(2\pi\eta_1 y_o)+\frac{1}{2}\cos(4\pi\eta_1 y_o)\right]$$

$$\left.\begin{aligned} I_{max} &= \frac{3}{2}+2+\frac{1}{2} = 4 \\ I_{min} &= 3/2-2+\frac{1}{2} = 0 \end{aligned}\right\rangle \quad V = 1$$



NA = numerical aperture

$\eta_C = \frac{NA}{\lambda}$

AT LOW SPATIAL FREQUENCIES, ALL POWER FROM THE DIFFRACTED ORDERS PASS THE STOP, PRODUCING $V=1$. THIS OBSERVATION CONTINUES, UNTIL JUST PAST η_c, WHERE THE DIFFRACTED ORDERS ARE BLOCKED, PRODUCING $V=0$.

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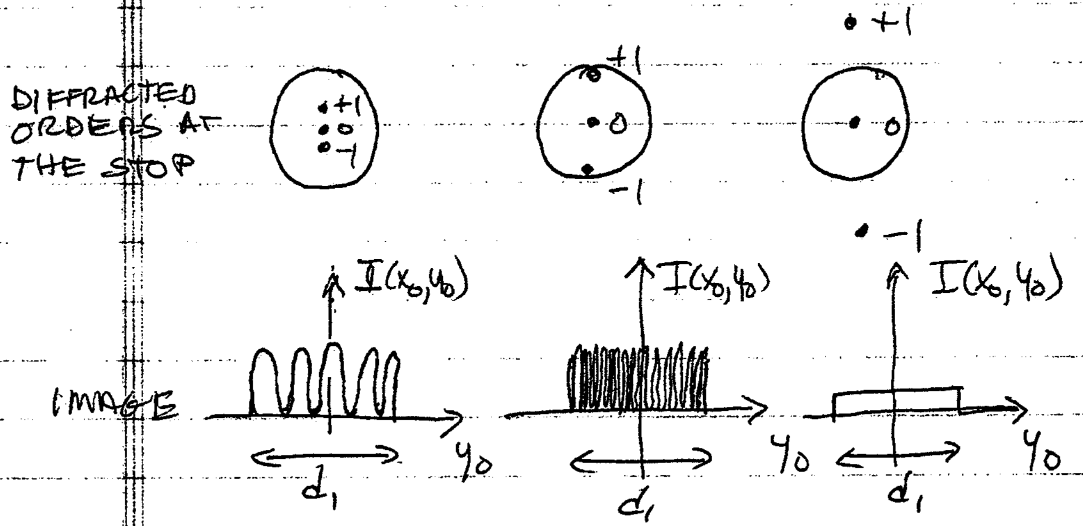
PHYSICAL INTERPRETATION OF $H(\xi,\eta)$: (CON'T)



LOW FREQ. HIGH V	JUST BELOW CUTOFF HIGH V	BEYOND CUTOFF V=0 BACKGROUND FROM ZERO ORDER

EXAMPLE 15.3:

$\lambda = 500\, nm$

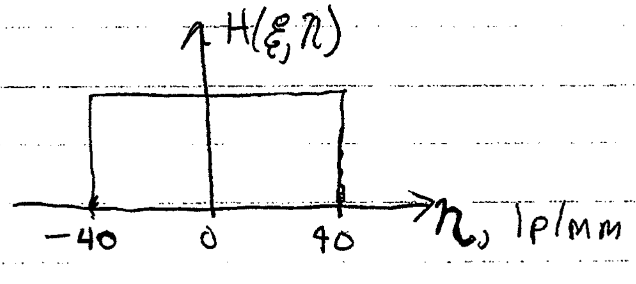
$f = 50\, mm$

$d_s = 2\, mm$

$$NA = \frac{d_s/2}{f} = \frac{1\times10^{-3}}{50\times10^{-3}} = 0.02$$

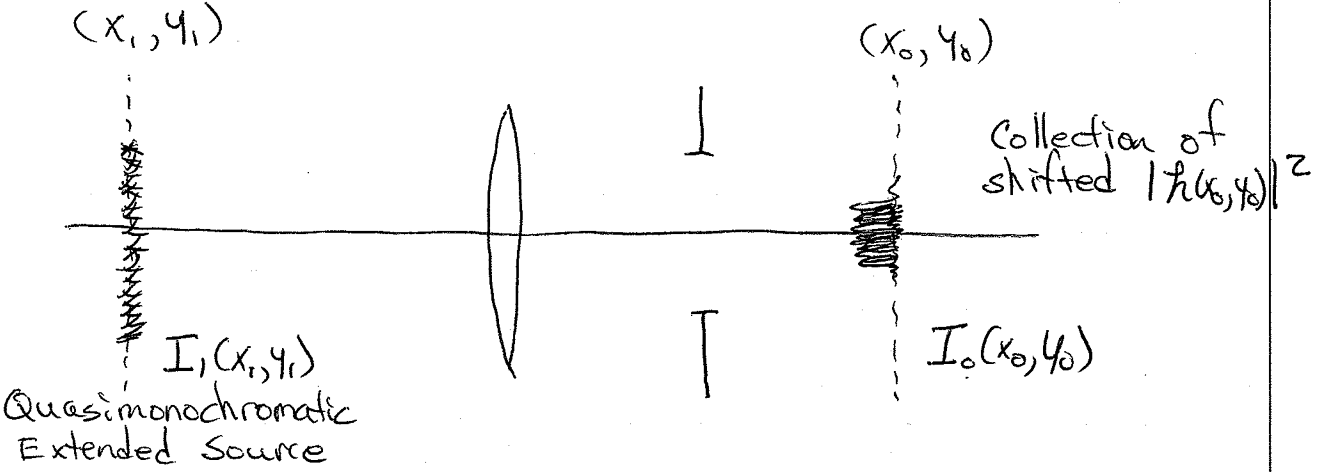
$$\eta_c = \frac{NA}{\lambda} = \frac{0.02}{500\times10^{-9}}\, m^{-1} = 4\times10^{4}\, m^{-1} = 40\, mm^{-1}$$

THESE UNITS ARE REFERRED TO AS "LINE-PAIRS PER mm" OR lp/mm



Incoherent Imaging:

Simple optical system with quasi monochromatic extended source:



Each point source produces an image point given by $|h(x_0-\Delta x_0, y_0-\Delta y_0)|^2$ and weighted by $I_1(x_1,y_1)$. Assuming linearity and shift invariance gives

$$I_0(x_0,y_0) = \overset{\substack{\text{Real}\\\text{constant}\\\downarrow}}{K}\; \underset{\substack{\uparrow\\\text{Scaled Geometrical}\\\text{Irradiance}}}{I_1\left(\frac{x_0}{m_T}, \frac{y_0}{m_T}\right)} ** \underset{\substack{\uparrow\\\text{Irradiance}\\\text{Point-Spread}\\\text{Function}\\\text{(iPSF)}}}{|h(x_0,y_0)|^2}$$

Frequency Analysis:

$$F_\eta F_\xi\left[I_o(x_o,y_o)\right] = K\, F_\eta F_\xi\left[I_i\left(\tfrac{x_o}{m_T},\tfrac{y_o}{m_T}\right)\right] F_\eta F_\xi\left[\left|h(x_o,y_o)\right|^2\right]$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$\mathcal{I}_o(\xi,\eta) = K m_T^2\, \mathcal{I}_i(m_T\xi, m_T\eta)\, \mathcal{H}(\xi,\eta)$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

(Spatial frequency distribution of image irradiance) = (Spatial frequency distribution of object radiant exitance) (Optical Transfer Function (OTF))

Optical Transfer Function (OTF):

$$\mathcal{H}(\xi,\eta) = F_\eta F_\xi\left[\left|h(x_o,y_o)\right|^2\right]$$

$$= F_\eta F_\xi\left[h(x_o,y_o)\right] \star\star\, F_\eta F_\xi\left[h^*(x_o,y_o)\right]$$

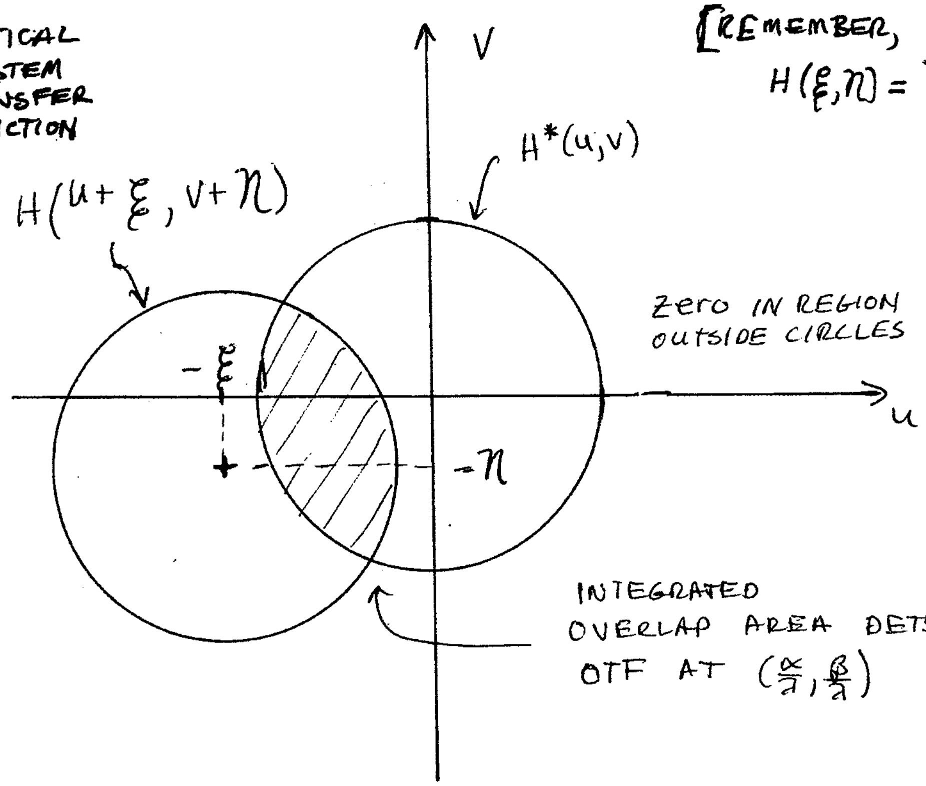
$$= H(\xi,\eta) \star\star H^*(\xi,\eta)$$

OTF(ξ,η) is the <u>complex</u> <u>autocorrellation</u> of CTF(ξ,η), which is just the scaled and reflected transmittance of the pupil!

$$\mathcal{H}(\xi,\eta) = \text{OTF}(\xi,\eta) = \text{constant} \cdot \iint_{-\infty}^{\infty} H(u+\xi, v+\eta)\, H^*(u,v)\, du\, dv$$

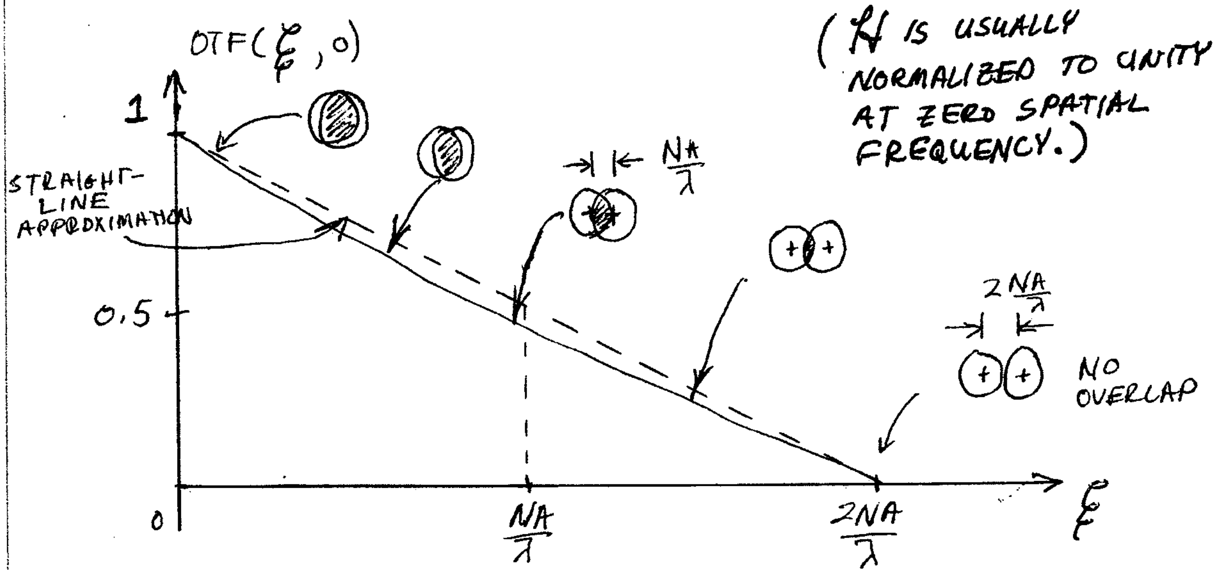
OTF = <u>O</u>PTICAL <u>S</u>YSTEM <u>T</u>RANSFER <u>F</u>UNCTION

[REMEMBER, $H(\xi,\eta) = T(x_s,y_s)\Big|_{\substack{x_s=-\lambda f\xi \\ y_s=-\lambda f\eta}}$]



ZERO IN REGION OUTSIDE CIRCLES

INTEGRATED OVERLAP AREA DETERMINES OTF AT $(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda})$



($\mathcal{H}$ IS USUALLY NORMALIZED TO UNITY AT ZERO SPATIAL FREQUENCY.)

THE LENS PASSES FREQUENCIES FROM THE OBJECT UP TO A MAXIMUM SPATIAL FREQUENCY OF

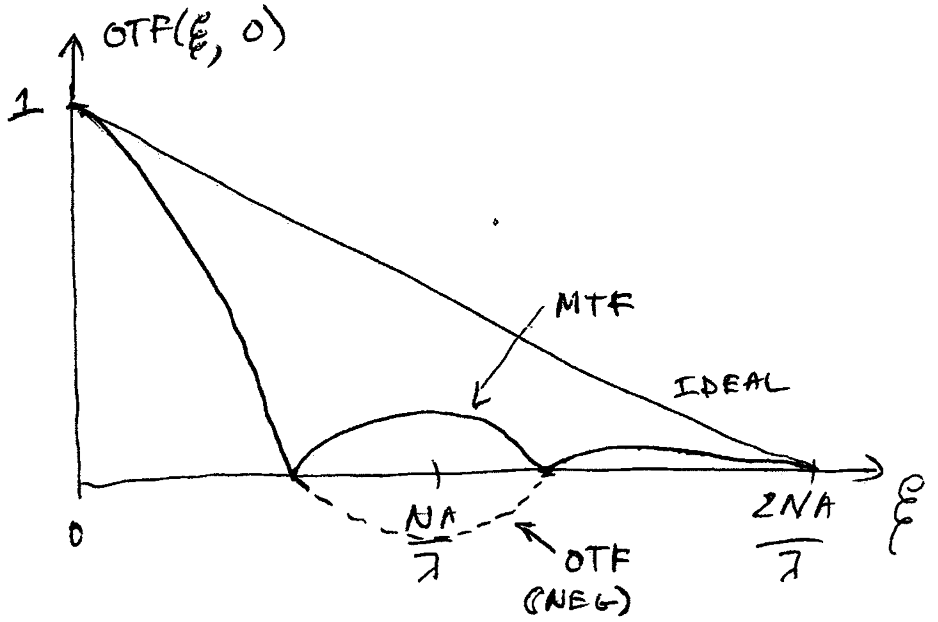
$$(\xi)_{MAX} = \frac{2NA}{\lambda} = \text{INCOHERENT CUTOFF}$$

$$= 2 \times \text{COHERENT CUTOFF}$$

MODULATION TRANSFER FUNCTION = MTF = $|OTF|$.

THE MTF IS USUALLY DISPLAYED.

WHAT HAPPENS WHEN $OTF(\xi,\eta)$ IS NEGATIVE?

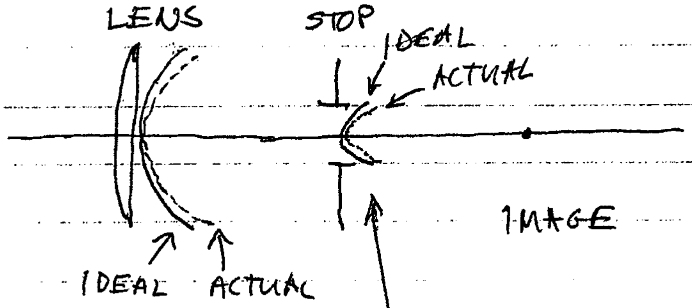


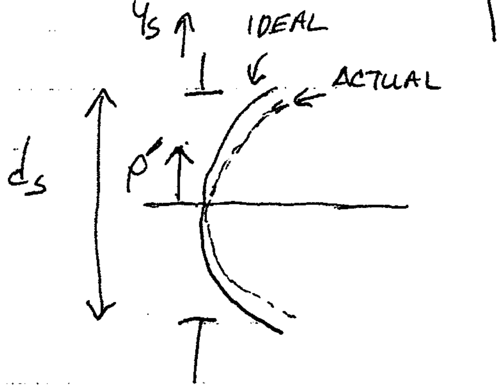
IN THE NEGATIVE REGION, CONTRAST REVERSAL IS OBSERVED.

THE EFFECTS OF ABERRATIONS ON $H(\xi,\eta)$:

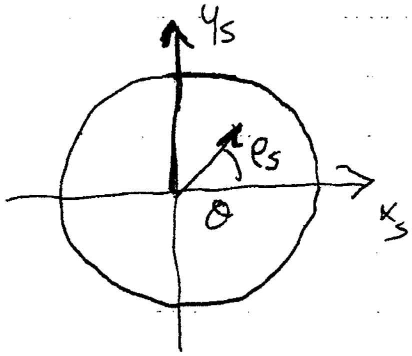
OFTEN, THE LENS DOES NOT PRODUCE AN IDEAL PHASE DISTRIBUTION $\phi(\rho_L) = -\frac{2\pi}{\lambda}\frac{\rho_L^2}{2f}$ WHEN LIGHT PASSES THROUGH IT.

THE IDEAL PHASE DISTRIBUTION IS A CONVERGING SPHERICAL WAVE. DIFFERENCES BETWEEN THE IDEAL CONVERGING SPHERE AND THE REAL PHASE DISTRIBUTION, AS MEASURED IN THE EXIT PUPIL, ARE THE ABERRATIONS OF THE SYSTEM.





$$\rho' = \frac{\rho_s}{d/2}$$



THE EFFECTS OF ABERRATIONS ON $H(\xi,\eta)$: (CON'T.)

ABERRATIONS ARE SPECIFIED IN TERMS OF THE ADDTIONAL OPTICAL PATH DIFFERENCE (OPD) BETWEEN A RAY PASSING THROUGH THE CENTER OF THE STOP AND ONE THROUGH $(\rho' \cos\theta, \rho' \sin\theta)$ THE RELATIVE FIELD DEPENDENCE IS H' IN THE IMAGE PLANE. ~~THE~~ COMMON ABERRATIONS ARE:

DEFOCUS : $W(\rho', \theta, H') = W_{020}\, \rho'^2$

COMA : $W(\rho', \theta, H') = W_{131} \cos\theta\, \rho^3 H'$

ASTIGMATISM : $W(\rho', \theta, H') = W_{222} \cos^2\theta\, \rho^2 H'^2$

SPHERICAL : $W(\rho', \theta, H') = W_{040}\, \rho'^4$

ADDITIONAL PHASE IS $\phi' = \frac{2\pi}{\lambda} W$, WHERE W IS IN UNITS OF LENGTH, OR $\phi' = 2\pi W$ IF W IS IN UNITS OF WAVELENGTH.

FOR THIS CLASS, WE WILL ASSUME THAT THE FIELD DEPENDENCE IS CONSTANT OVER THE AREA OF INTEREST IN THE IMAGE PLANE. THIS CONDITION IS CALLED ISOPLANICITY, AND THE REGION OF OBSERVATION IS AN ISOPLANATIC PATCH.

THE EFFECT OF ABERRATIONS:

OFTEN, LENSES DO NOT FORM PERFECT IMAGES. DUE TO ABERRATIONS IN THE OPTICAL SYSTEM. THESE ABERRATIONS ARE CHARACTERIZED IN THE STOP BY A FUNCTION $W(x_s, y_s)$ THAT DESCRIBES THE OPTICAL PATH DIFFERENCE (IN WAVELENGTH UNITS) OF LIGHT AT (x_s, y_s) RELATIVE TO LIGHT PASSING THROUGH THE SYSTEM AT $(0,0)$. TO ACCOUNT FOR ABERRATIONS IN THE OTF, WE SIMPLY MULTIPLY THE STOP TRANSMISSION $T(x_s,y_s)$ BY THE PHASE SHIFT INTRODUCED BY $W(x_s, y_s)$. THAT IS,

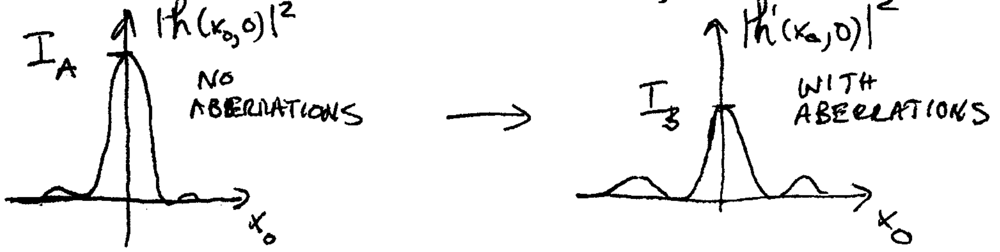
$$T'(x_s, y_s) = T(x_s, y_s)\, e^{j2\pi W(x_s,y_s)} .$$

$\mathcal{H}'(\xi,\eta)$ IS NOW THE AUTOCORRELATION OF THE MODIFIED (AND APPROPRIATELY SCALED) STOP TRANSMISSION FUNCTION T'. THE NET EFFECT OF ABERRATIONS IS TO DECREASE THE VALUE OF THE TRANSFER FUNCTION IN CERTAIN FREQUENCY RANGES, DEPENDING ON THE TYPE OF ABERRATION. (SEE CHARTS FOR SOME EXAMPLES.)

NOTICE THAT $$h'(x_o, y_o) = \mathcal{F}_{\frac{y_o}{\lambda f}} \mathcal{F}_{\frac{x_o}{\lambda f}} \left[H'(\xi,\eta) \right]$$

Modified and scaled stop transmission function is now $H'(\xi,\eta)$

MUST NOW INCLUDE THE EFFECTS OF ABERRATIONS. FOR EXAMPLE,



THE STREHL RATIO IS DEFINED BY

$$SR = \frac{I_B}{I_A},$$ WHERE I_A AND I_B ARE IMAGE IRRADIANCE VALUES AT $(0,0)$ WITHOUT AND WITH ABERRATIONS, RESPECTIVELY.

RELATIONSHIP BETWEEN SR AND $\mathcal{H}$:

$$SR = \frac{\left|h'(0,0)\right|^2_{\text{WITH ABS}}}{\left|h(0,0)\right|^2_{\text{WITHOUT ABS}}}$$

NOTE THAT

$$\left|h'(x_0,y_0)\right|^2 = F^{-1}_{\frac{y_0}{\lambda f}} F^{-1}_{\frac{x_0}{\lambda f}} \left[\mathcal{H}'(\xi,\eta)\right]$$

$$= \iint_{-\infty}^{\infty} \mathcal{H}'(\xi,\eta)\, e^{j2\pi\left(\xi\frac{x_0}{\lambda f} + \eta\frac{y_0}{\lambda f}\right)} d\xi\, d\eta$$

SO

$$\left|h'(0,0)\right|^2 = \iint_{-\infty}^{\infty} \mathcal{H}'(\xi,\eta)\, d\xi\, d\eta$$

(THIS IS JUST THE VOLUME UNDER THE ABERRATED OTF.)

AND

$$\left|h(0,0)\right|^2 = \iint_{-\infty}^{\infty} \mathcal{H}(\xi,\eta)\, d\xi\, d\eta$$

(AREA UNDER UNABERRATED OTF)

THEREFORE,

$$SR = \frac{\iint_{-\infty}^{\infty} \mathcal{H}'(\xi,\eta)\, d\xi\, d\eta}{\iint_{-\infty}^{\infty} \mathcal{H}(\xi,\eta)\, d\xi\, d\eta} = \frac{\text{AREA UNDER ABERRATED OTF}}{\text{AREA UNDER UNABERRATED OTF}}$$

SMALL ABERRATIONS $\Rightarrow$

$$\left|h'(0,0)\right|^2 = \left.\left|h'(x_0,y_0)\right|^2\right|_{x_0=y_0=0} = \left|F_{\frac{y_0}{\lambda f}} F_{\frac{x_0}{\lambda f}}\left[H(\xi,\eta)\right]\right|^2_{x_0=y_0=0}$$

$$= \left|\iint_{-\infty}^{\infty} T(-\lambda f\xi, -\lambda f\eta)\, e^{-j2\pi W(-\lambda f\xi, -\lambda f\eta)} d\xi\, d\eta\right|^2$$

$$\approx \left|\iint_{\text{Stop}} \left[1 - j2\pi W(-\lambda f\xi, -\lambda f\eta) - \frac{1}{2}(2\pi)^2 W^2(-\lambda f\xi, -\lambda f\eta)\right] d\xi\, d\eta\right|^2$$

(CON'T.)

$$I_B = |h'(0,0)|^2 \approx \left| \iint_{stop} d\xi d\eta - j2\pi \iint_{STOP} W(-f\lambda\xi, -f\lambda\eta) d\xi d\eta - \frac{1}{2}(2\pi)^2 \iint_{stop} W^2(-\lambda f\xi, -\lambda f\eta) d\xi d\eta \right|^2$$

$$= \left| \iint_{stop} d\xi d\eta \right|^2 \left| 1 - j2\pi \frac{\iint_{STOP} W(-f\lambda\xi, -f\lambda\eta) d\xi d\eta}{\iint_{stop} d\xi d\eta} - \frac{1}{2}(2\pi)^2 \frac{\iint_{STOP} W^2(-f\lambda\xi, -f\lambda\eta) d\xi d\eta}{\iint_{stop} d\xi d\eta} \right|^2$$

$$= \left| \iint_{stop} d\xi d\eta \right|^2 \left| 1 - j2\pi\overline{W} - \frac{1}{2}(2\pi)\overline{W^2} \right|^2$$

$$I_A = |h(0,0)|^2 = \left| \iint_{stop} d\xi d\eta \right|^2$$

so,

$$SR = \frac{I_B}{I_A} \approx \left| 1 - j2\pi\overline{W} - \frac{1}{2}(2\pi)^2\overline{W^2} \right|^2$$

$$= \left(1 - j2\pi\overline{W} - \frac{1}{2}(2\pi)^2\overline{W^2}\right)\left(1 + j2\pi\overline{W} - \frac{1}{2}(2\pi)^2\overline{W^2}\right)$$

$$= 1 + \cancel{j2\pi\overline{W}} - 2\pi^2\overline{W^2} - \overset{+4\pi^2(\overline{W})^2}{\cancel{j2\pi\overline{W}}\,|\, + j2\pi\overline{W}}\,\cancel{2\pi^2}\overline{W^2}$$
$$- 2\pi^2\overline{W^2} - j2\pi^2\cancel{\overline{W^2}\,2\pi\overline{W}} + 4\pi^4\cancel{\left(\overline{W^2}\right)^2}^{\sim 0}$$

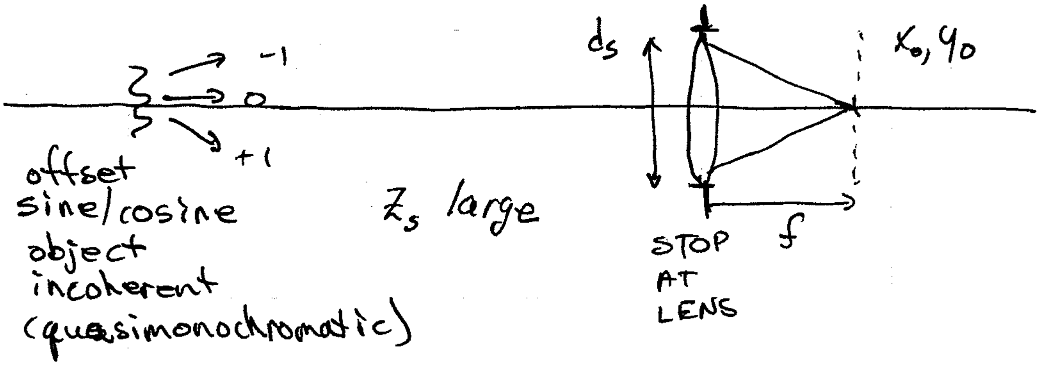
$$\approx 1 - 4\pi^2\left[\overline{W^2} - (\overline{W})^2\right]$$

$$= 1 - 4\pi^2\sigma^2$$

↑ WAVEFRONT VARIANCE.

INCOHERENT TF ADD'L NOTES.

DISTANT OBJECT W/ STOP AT LENS.



$NA = \frac{d_s}{2f}$

TF analysis works for the entrance pupil in the Fraunhofer zone of the object and the image in the Fraunhofer zone of the exit pupil.

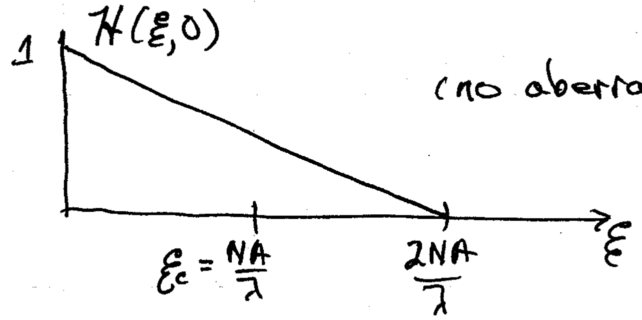
For the stop at the lens:

$$|h(x_0,y_0)|^2 = \left| K\, F_{\eta=\frac{y_0}{\lambda f}}\, F_{\xi=\frac{x_0}{\lambda f}} \left[\mathrm{circ}\left(\frac{x_s^2+y_s^2}{d_s} \right) \right] \right|^2$$

$$= \left(\frac{K\pi d_s^2}{4} \right)^2 \mathrm{somb}^2 \left(\frac{d_s}{\lambda f} \sqrt{x_0^2+y_0^2} \right),$$

which is the <u>same</u> result as with the stop at the back focus.

$$\therefore\ \mathcal{H}(\xi,\eta) = OTF(\xi,\eta) = T(-\lambda f\xi, -\lambda f\eta) \star\star T^*(-\lambda f\xi, -\lambda f\eta)$$



<u>Example</u>: $\lambda = 500\,nm$ $\quad NA = \frac{d}{2f} = 0.1$

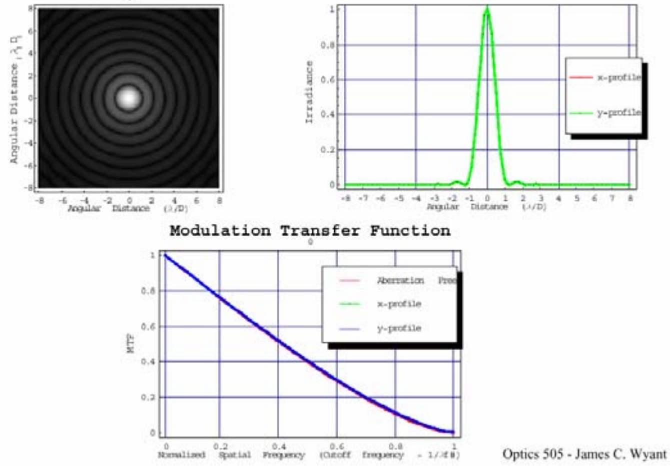
$d_s = 10\,mm$ $\quad 2\xi_c = \frac{2NA}{\lambda} = 200{,}000\ m^{-1} = 200\ lp/mm$

$f = 50\,mm$

specified as "line pairs per mm"

In object space, $\Delta y_0 = 1mm$ corresponds to $\frac{1}{50}$ rad.

Aberration Free



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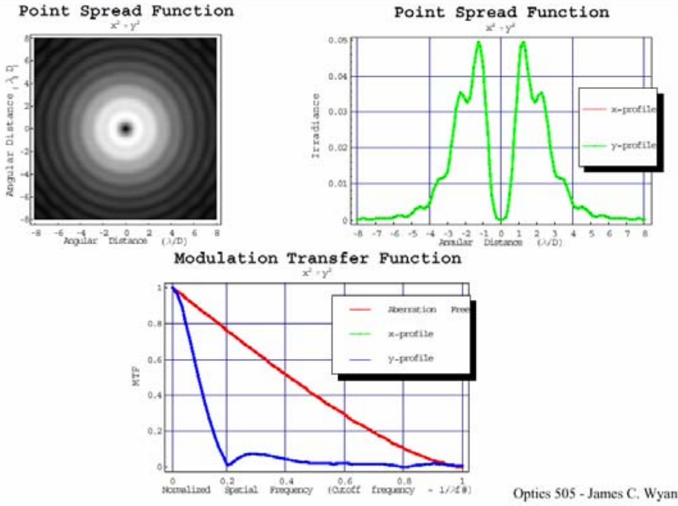
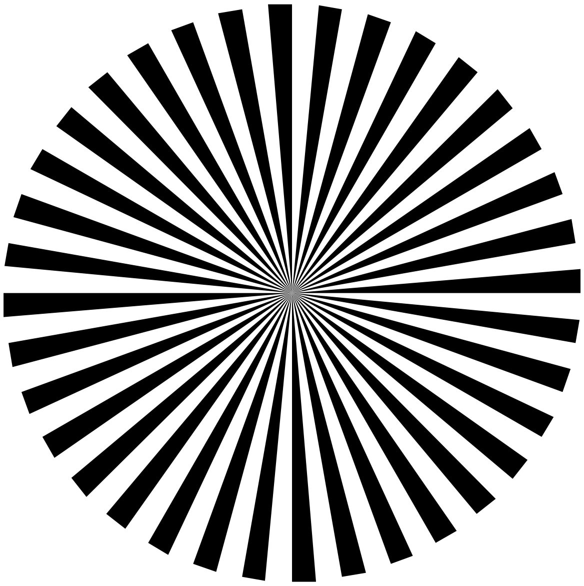
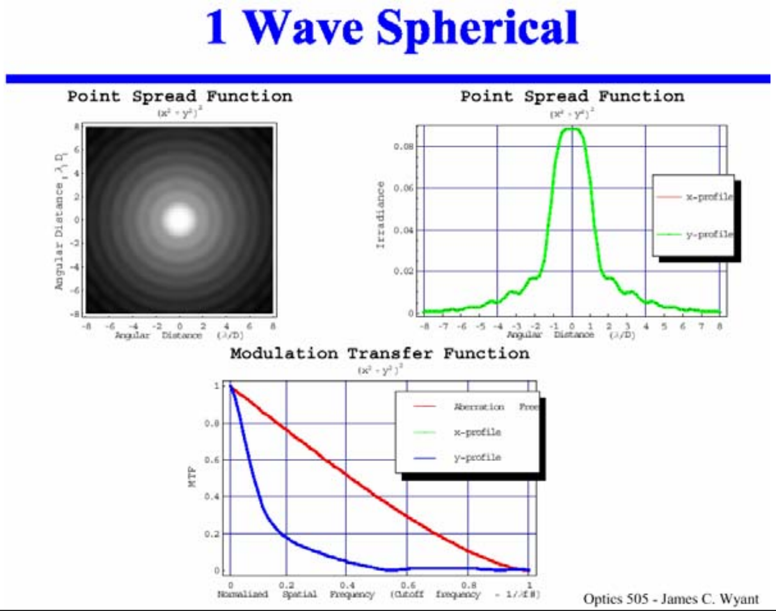
1 Wave Defocus



Optics 505 - James C. Wyant





1 Wave Spherical

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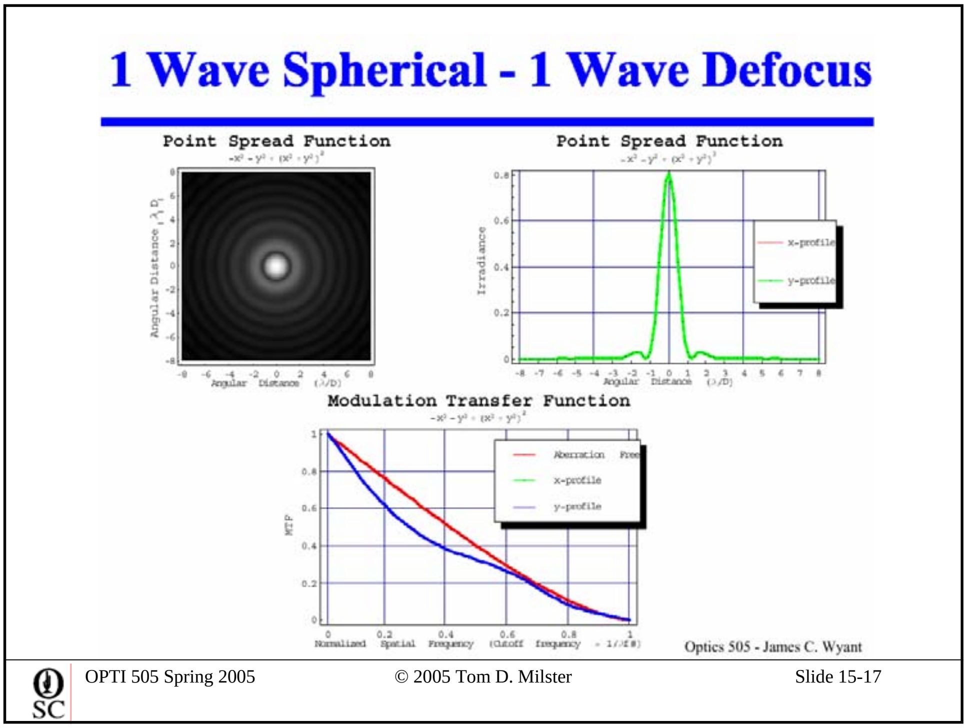

1 Wave Spherical - 1 Wave Defocus

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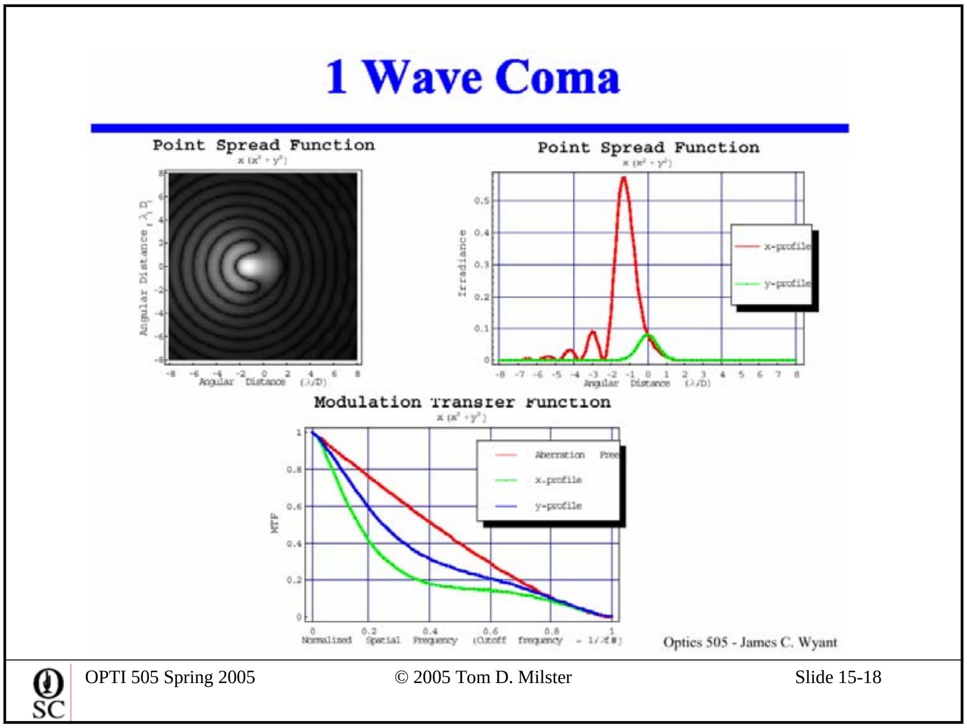

1 Wave Coma

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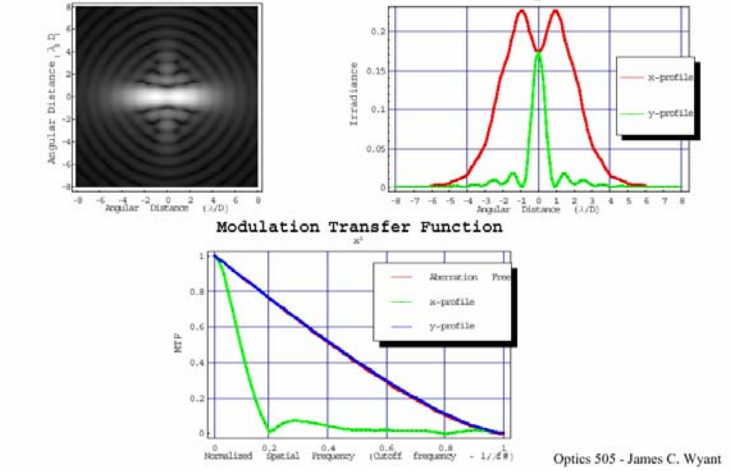
1 Wave Astigmatism



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