Solutions to the Wave Equation (Part A)

Diffraction and Interferometry

Simple Transverse Waves (I)

One-dimensional wave equation:

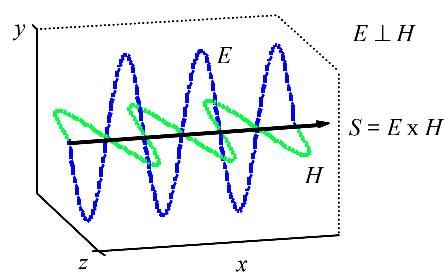
$$\frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} = 0$$

Solution traveling in $+x$ direction:

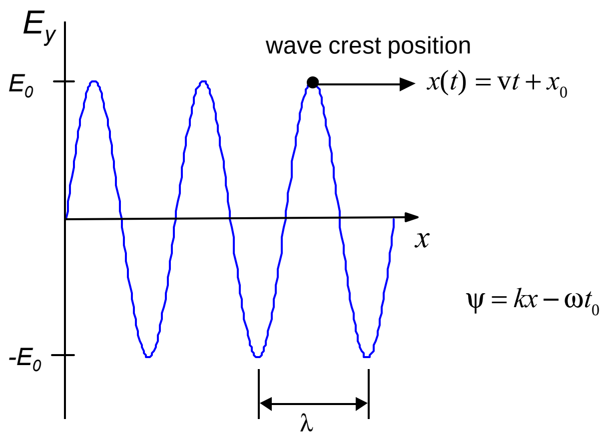
$$\vec{E}_y(x,t) = E_0 \cos(kx - \omega t)\,\hat{y}$$

if $k = \frac{\omega}{c}$.

The complete solution involves consideration of the magnetic field. If we take a snapshot in time of the EM wave it may look like:



Simple Transverse Waves (II)



Plane Waves

Three-dimensional wave equation in free space:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Solution traveling in direction $\hat{k}$:

$$\vec{E}(\vec{r}) = \vec{E}_0 e^{j(\vec{k}\bullet\vec{r} - \omega t)}$$

where $\vec{r} = r\hat{r} = x\hat{x} + y\hat{y} + z\hat{z}$

and $\vec{k} = k\hat{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} = k(\alpha\hat{x} + \beta\hat{y} + \gamma\hat{z})$.

Note that $\vec{k}\bullet\vec{r} - \omega t = k_x x + k_y y + k_z z - \omega t =$ constant

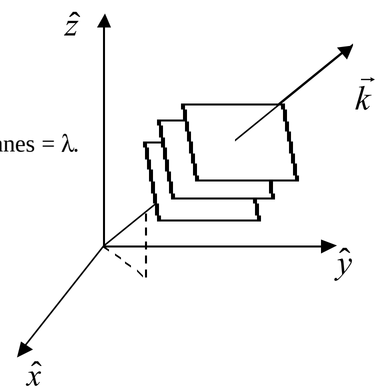
represents planes in space of constant phase. Hence, we call this solution a *plane wave*.

Scalar Plane Waves

$E(\vec{r}) = E_0 e^{j(\vec{k}\bullet\vec{r}-\omega t)}$

Wave crests $\perp \vec{k}$.

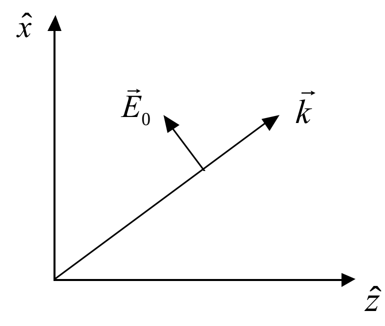
Max separation between planes = λ.



Vector EM Plane Waves

$\vec{E}(\vec{r}) = \vec{E}_0 e^{j(\vec{k}\bullet\vec{r}-\omega t)}$

$\vec{E}_0 \perp \vec{k}$



Scalar Spherical Waves

A solution to the wave equation in spherical coordinates is:

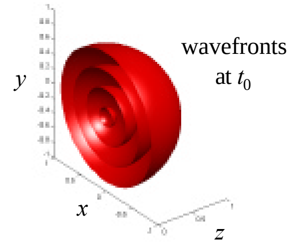
$$E(r,t) = \frac{1}{r} e^{j(kr-\omega t)} \quad ,$$

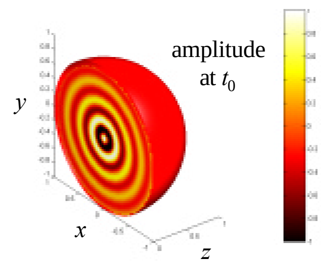
where

$$r = \sqrt{x^2 + y^2 + z^2}$$

and $\psi = kr - \omega t$.

Wave crests (also called *wavefronts*) at ψ = constant expand outwardly from the source point as a function of time. Wavefronts are perpendicular to r.





Vector Spherical Waves

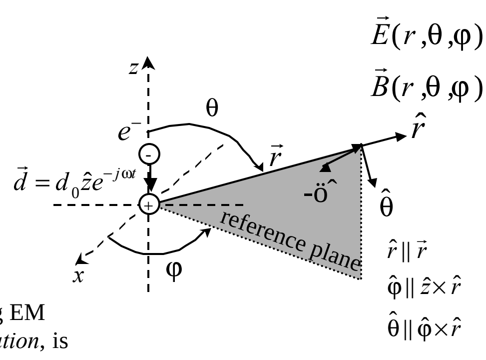
In the wave zone, where $r \gg \frac{\lambda}{2\pi}$,

$$E_r \approx 0$$

$$E_\theta \approx \frac{e d_0 k^2 \sin\theta}{4\pi\varepsilon_0 r} e^{j(kr-\omega t)}$$

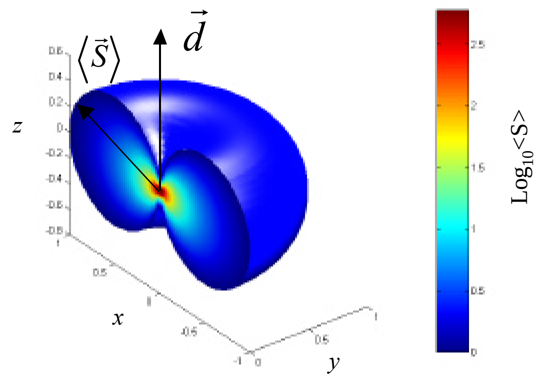
$$B_\varphi \approx \frac{c\mu_0 d_0 k^2 \sin\theta}{4\pi r} e^{j(kr-\omega t)}$$

This configuration of time-varying EM fields, which is called *dipole radiation*, is very important in understanding the physical interpretation of wave optics.



Vector Spherical Waves (Power Flow)

Although the wavefronts for a dipole radiator in the wave zone are the same as a scalar spherical wave, the EM radiation is amplitude modulated as a function of θ.



Linearity

Since the electric field is a vector quantity, a vector summation must be performed to find the electric field resulting from the summation of several electric fields.

$$\vec{E}_{total} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots$$

This linear superposition is only approximately true in the presence of matter. Deviations from linearity are observed at high intensities produced by lasers when the electric fields approach the electric fields comparable to atomic fields (non-linear optics). In these notes we will consider only situations where linear superposition is valid.

Waves Having Same Frequency, but Different Amplitude and Phase

$$E(x,t) = A_0 e^{j\phi_0} e^{j(kx-\omega t)} = A_1 e^{j\phi_1} e^{j(kx-\omega t)} + A_2 e^{j\phi_2} e^{j(kx-\omega t)}$$

$$A_0 e^{j\phi_0} = A_1 e^{j\phi_1} + A_2 e^{j\phi_2}$$

$$= A_1 \cos\phi_1 + A_2 \cos\phi_2 + j\left\{ A_1 \sin\phi_1 + A_2 \sin\phi_2 \right\}$$

$$\tan\phi_0 = \frac{A_0 \sin\phi_0}{A_0 \cos\phi_0} = \frac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi_1 + A_2 \cos\phi_2}$$

$$I_0 \propto A_0^2 = \left| A_1 e^{j\phi_1} + A_2 e^{j\phi_2} \right|^2 = \boxed{A_1^2 + A_2^2 + 2A_1A_2 \cos\left\{\phi_2 - \phi_1\right\}}$$

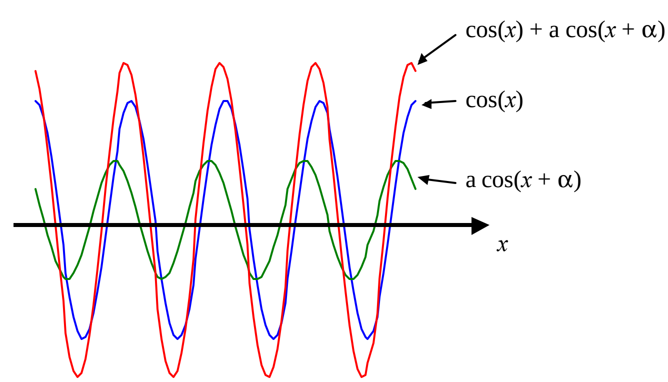
This is an important result that is the basis for much of the class.

If $A_1 = A_2$,

$$I_0 \propto A_0^2 = 2A_1^2 \left\{1 + \cos\left(\phi_2 - \phi_1\right)\right\} = 4A_1^2 \cos^2\left\{\frac{1}{2}\left(\phi_2 - \phi_1\right)\right\}$$

Linear wave combination gives a cosine irradiance distribution.

Waves Having Same Frequency, but Different Amplitude and Phase



Linear combination is a wave of the same frequency.

Combination of Several Waves Having Same Frequency, but Different Amplitude and Phase

$$E(x,t) = A_0 e^{j\phi_0} e^{j(kx-\omega t)} = \sum_{n=1}^{N} A_n e^{j\phi_n} e^{j(kx-\omega t)}$$

$$A_0 e^{j\phi_0} = \sum_{n=1}^{N} A_n e^{j\phi_n}$$

$$\tan\phi_0 = \frac{\sum_{n=1}^{N} A_n \sin\phi_n}{\sum_{n=1}^{N} A_n \cos\phi_n}$$

$$A_0^2 = \sum_{n=1}^{N} A_n^2 + \sum_{n=1}^{N} \sum_{m=1,m<n}^{N} 2A_nA_m\cos(\phi_n - \phi_m)$$

interference term

Combination of Several Waves Having Same Frequency, but Different Amplitude and Phase

For ϕ_n's random over the observation time, $A_0^2 = \sum_{n=1}^{N} A_n^2$.

$A_0^2 = NA_n^2$ if all A_n are equal. This is generally referred to as *incoherent* addition.

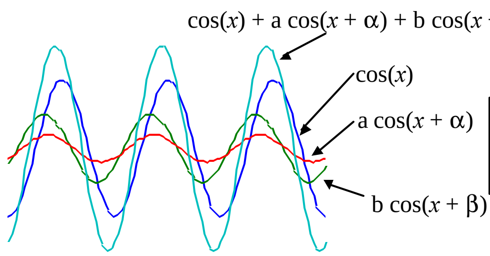
If $\phi_n =$ constant over the observation time,

$A_0^2 = \left(\sum_{n=1}^{N} A_n\right)^2$, which is called *coherent* addition.

If all A_n are equal, $A_0^2 = N^2A_n^2$ for coherent addition.

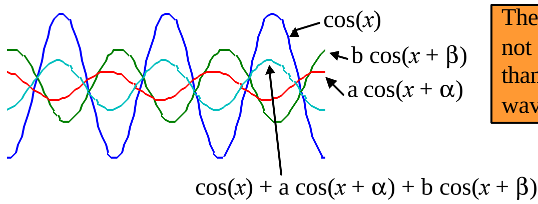
Combination of Several Waves Having Same Frequency, but Different Amplitude and Phase

Three waves with same frequency (case 1):



Linear addition produces a wave with the same frequency.

Three waves with same frequency (case 2):



The resulting wave is not always larger than the component waves.

Beats

Consider two waves with the same amplitude, but different frequency.

$$E_1(x,t) = A\cos(k_1x - \omega_1 t + \phi_1)$$

$$E_2(x,t) = A\cos(k_2x - \omega_2 t + \phi_2)$$

Adding the two waves produces

$$E(x,t) = A\left\{\cos(k_1x - \omega_1 t + \phi_1) + \cos(k_2x - \omega_2 t + \phi_2)\right\} \;.$$

Use of the identity $\cos U + \cos V = 2\cos\left(\frac{U+V}{2}\right)\cos\left(\frac{U-V}{2}\right)$ yields

$$E(x,t) = 2A\left[\cos\left\{\left(\frac{k_1+k_2}{2}\right)x - \left(\frac{\omega_1+\omega_2}{2}\right)t + \left(\frac{\phi_1+\phi_2}{2}\right)\right\} \cdots\right.$$

$$\left.\cos\left\{\left(\frac{k_1-k_2}{2}\right)x - \left(\frac{\omega_1-\omega_2}{2}\right)t + \left(\frac{\phi_1-\phi_2}{2}\right)\right\}\right] \;.$$

Beats

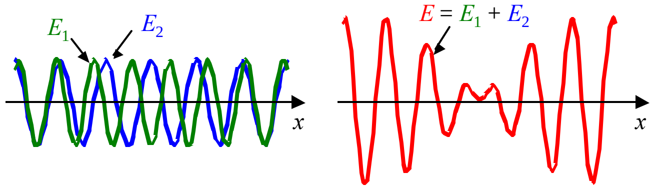
Let

$\omega_{HF} = \frac{1}{2}(\omega_1 + \omega_2)$ $\quad k_{HF} = \frac{1}{2}(k_1 + k_2)$ — high temporal frequency, small λ (HF)

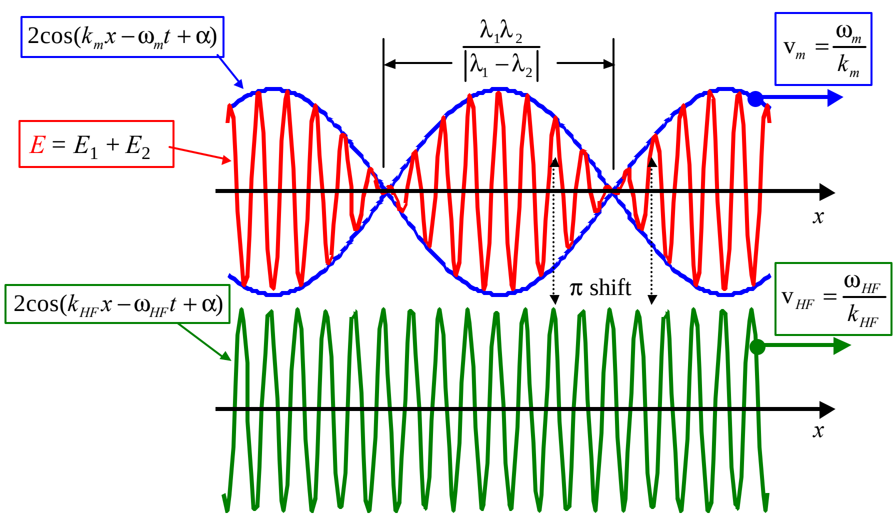
$\omega_m = \frac{1}{2}(\omega_1 - \omega_2)$ $\quad k_m = \frac{1}{2}(k_1 - k_2)$ — lower temporal frequency, larger λ (modulation)

$\alpha = \frac{1}{2}(\phi_1 + \phi_2) \qquad \beta = \frac{1}{2}(\phi_1 - \phi_2)$

$$E(x,t) = 2A\cos(k_{HF}x - \omega_{HF}t + \alpha)\cos(k_m x - \omega_m t + \beta)$$

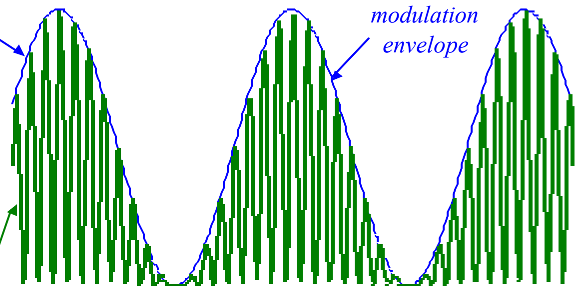


Beats



Beats - Irradiance

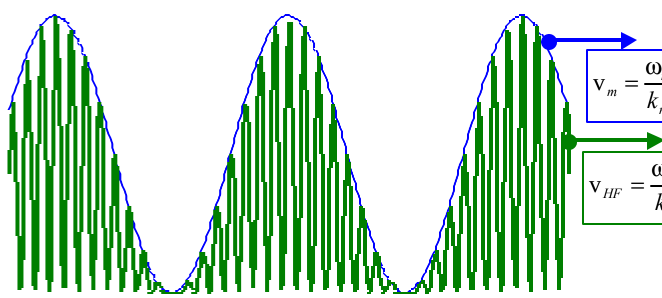
$4A^2\cos^2(k_m x - \omega_m t + \beta)$



$$\lambda_{eq} = \frac{\lambda_1 \lambda_2}{|\lambda_1 - \lambda_2|}$$

$$E^2(x,t) = 2A^2\cos^2(k_{HF}x - \omega_{HF}t + \alpha)\left[1 + \cos\{2(k_m x - \omega_m t + \beta)\}\right]$$

Beats – Nondispersive Medium



$$\mathrm{v}_m = \frac{\omega_m}{k_m} = c$$

$$\mathrm{v}_{HF} = \frac{\omega_{HF}}{k_{HF}} = c$$

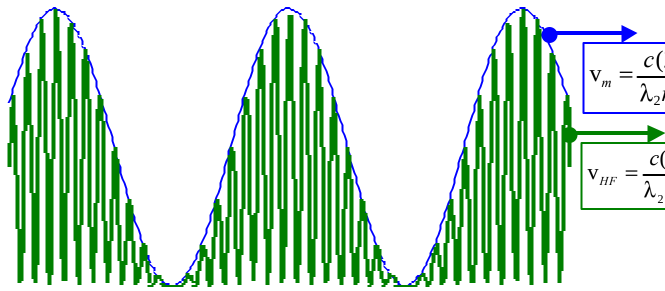
In a nondispersive medium, like air, $\frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = \mathrm{v} = c$.

$$k_{HF} = \frac{1}{2}(k_1 + k_2) \qquad k_m = \frac{1}{2}(k_1 - k_2)$$

$$\omega_{HF} = \frac{1}{2}(\omega_1 + \omega_2) = \frac{c}{2}(k_1 + k_2) \qquad \omega_m = \frac{1}{2}(\omega_1 - \omega_2) = \frac{c}{2}(k_1 - k_2)$$

Group v_m and phase v_{HF} velocities are equal for waves traveling in the same direction.

Beats – Dispersive Medium



$$v_m = \frac{c(\lambda_2 - \lambda_1)}{\lambda_2 n_1 - \lambda_1 n_2}$$

$$v_{HF} = \frac{c(\lambda_1 + \lambda_2)}{\lambda_2 n_1 + \lambda_1 n_2}$$

In a dispersive medium, like glass, $\frac{\omega_1}{k_1} = v_1 = \frac{c}{n_1}$ and $\frac{\omega_2}{k_2} = v_2 = \frac{c}{n_2}$.

$$k_{HF} = \frac{1}{2}(k_1 + k_2) \qquad k_m = \frac{1}{2}(k_1 - k_2)$$

$$\omega_{HF} = \frac{1}{2}(\omega_1 + \omega_2) = \frac{c}{2}\left(\frac{k_1}{n_1} + \frac{k_2}{n_2}\right) \qquad \omega_m = \frac{1}{2}(\omega_1 - \omega_2) = \frac{c}{2}\left(\frac{k_1}{n_1} - \frac{k_2}{n_2}\right)$$

Group v_m and phase v_{HF} velocities are not equal for waves traveling in the same direction.

Standing Waves

(Waves traveling in opposite directions)

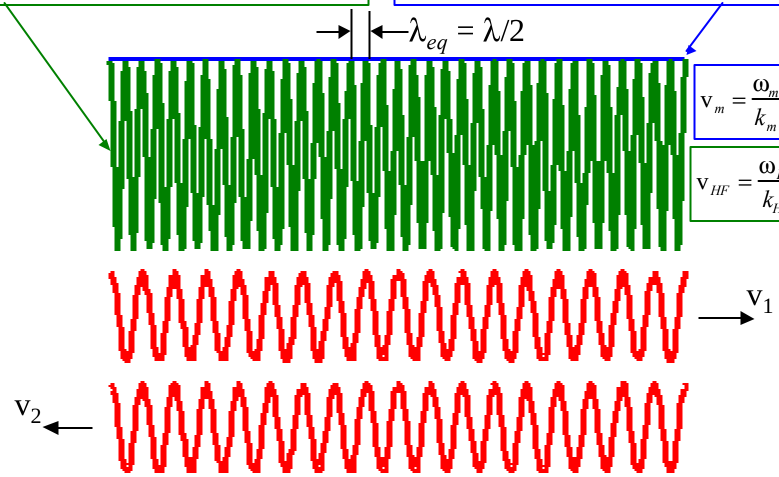
$$E_1(x,t) = A\cos(kx - \omega t + \phi_1)$$
$$E_2(x,t) = A\cos(kx + \omega t + \phi_2)$$
$$k_1 = k_2 = k$$
$$\omega_1 = -\omega_2 = \omega$$
$$E(x,t) = 2A\cos(k_{HF}x - \omega_{HF}t + \alpha)\cos(k_m x - \omega_m t + \beta)$$
$$= 2A\cos(kx + \alpha)\cos(-\omega t + \beta)$$
$$I(x) = 4A^2\cos^2(kx + \alpha) = 2A^2[1 + \cos(2kx + 2\alpha)]$$

$$\lambda_{eq} = \lambda/2$$

Standing Waves

$4A^2\cos^2(k_{HF}x-\omega_{HF}t+\alpha)=4A^2\cos^2(kx+\alpha)$

$4A^2\cos^2(k_m x-\omega_m t+\beta)=4A^2\cos^2(-\omega t+\beta)$



$\mathrm{v}_m=\dfrac{\omega_m}{k_m}=\infty$

$\mathrm{v}_{HF}=\dfrac{\omega_{HF}}{k_{HF}}=0$