OPTI 507 - Solid-State Optics
HW 4 Solutions

Jason M. Auxier

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Homework 4 consists of problems 3.2 - 3.5 in the text: *Introduction to Semiconductor Optics* by Peyghambarian *et al* [1].

1 Problem 3.2 - 10 Points

Prove Eq. (3.61).

1.1 Solution

Consider a prism with index $n > 1$. Looking at Fig. 1, we see that the input beam is headed to the normal of the output face. Let $\gamma' = \pi/2 - \gamma$, where γ is the incident angle. By Snell's law,

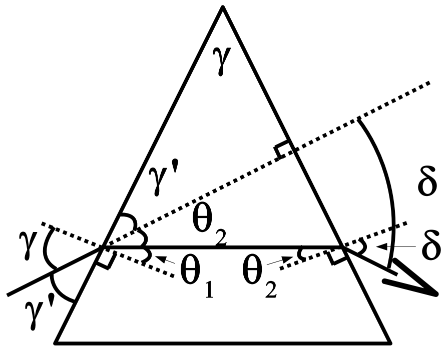

Figure 1: The geometry of the prism along with the angle definitions.

$$\begin{aligned} \sin\gamma &= n\sin\theta_1 \\ n\sin\theta_2 &= \sin\delta \end{aligned} \tag{1}$$

Note that $\gamma' + \theta_1 + \theta_2 = \pi/2$, so

$$\gamma = \theta_1 + \theta_2 \tag{2}$$

Thus, using a trig identity, we can write

$$\sin\gamma = \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2. \tag{3}$$

Also,

$$\begin{aligned} \cos\theta_2 &= \sin(\gamma' + \theta_1) = \cos\gamma' \sin\theta_1 + \sin\gamma' \cos\theta_1 \\ &= \sin\gamma \sin\theta_1 + \cos\gamma\cos\theta_1. \end{aligned} \tag{4}$$

Now, using Eq. 4, $\sin\theta_1 = \frac{\sin\gamma}{n}$, $\sin\theta_2 = \frac{\sin\delta}{n}$, and $\cos\alpha = \sqrt{1-\sin^2\alpha}$, we can write Eq. 3 as

$$\sin\gamma = \frac{1}{n}\sqrt{1-\left(\frac{\sin\delta}{n}\right)^2}\left(\sin\delta + \sin\gamma\cos\gamma\right) + \frac{1}{n^2}\sin^3\gamma. \tag{5}$$

Multiplying both sides of this equation by n^2 and bringing $\sin^3\gamma$ to the LHS, we have

$$\sin\gamma\left(n^2 - \sin^2\gamma\right) = \sqrt{n^2-\sin^2\delta}\left(\sin\delta + \sin\gamma\cos\gamma\right). \tag{6}$$

Squaring both sides, bringing everything to the LHS, and factoring, we can write

$$\left(n^2-\sin^2\gamma\right)\left(n^2\sin^2\delta - \sin^4\delta - \sin^2\delta - 2\sin\delta\sin\gamma\cos\gamma - \sin^2\gamma\cos^2\gamma\right) = 0. \tag{7}$$

Now, $n = \pm\sin\gamma$ is nonphysical, so the second term must vanish:

$$\boxed{\begin{aligned} n^2 &= \frac{\sin^2\delta\left(1-\cos^2\delta\right) + \sin^2\delta + 2\sin\delta\sin\gamma\cos\gamma + \sin^2\gamma\cos^2\gamma}{\sin^2\gamma} \\ &= 1 + \frac{2\cos\gamma\sin\delta}{\sin\gamma} + \frac{\sin^2\delta}{\sin^2\gamma}. \end{aligned}} \tag{8}$$

2 Problem 3.3 - 10 Points

Show that the reflectivity of a material with negative dielectric function approaches unity.

2.1 Solution

For a negative dielectric function, $\epsilon'' = 0$ and $\epsilon' = -|\epsilon'|$, so Eq. (3.18) in the text [1] becomes

$$\begin{aligned} n(\omega) &= \sqrt{\frac{\epsilon' + |\epsilon'|}{2}} \\ &= 0. \end{aligned} \tag{9}$$

So, putting $n = 0$ into the reflectance equation:

$$R(\omega) = \frac{(n(\omega) - 1)^2 + \left(\frac{\alpha c}{2\omega}\right)^2}{(n(\omega) + 1)^2 + \left(\frac{\alpha c}{2\omega}\right)^2}, \tag{10}$$

we see that $R \to 1$. Notice that α can have any arbitrary, finite value.

3 Problem 3.4 - 10 Points

Write the field components of a TE surface mode for the metal-vacuum interface, and prove that the interface cannot support a TE mode.

3.1 Solution

For the TE mode, the electric field is perpendicular to the plane of incidence. If the plane of incidence is the x-z plane, then the electric field is in the y-direction. Denoting medium 1 as the region $z > 0$ and medium 2 as the region $z < 0$, we have

$$\begin{aligned} \mathbf{E_1}(\mathbf{r}) &= (0, E_1, 0)e^{i(k_x x - \xi t)}e^{-\alpha_1 t} \text{ for } z > 0 \\ \mathbf{E_2}(\mathbf{r}) &= (0, E_2, 0)e^{i(k_x x - \xi t)}e^{\alpha_2 t} \text{ for } z < 0 \\ \mathbf{H_1}(\mathbf{r}) &= (H_{1x}, 0, H_{1z})e^{i(k_x x - \xi t)}e^{-\alpha_1 t} \text{ for } z > 0 \\ \mathbf{H_2}(\mathbf{r}) &= H_{1x}, 0, H_{1z})e^{i(k_x x - \xi t)}e^{\alpha_2 t} \text{ for } z < 0, \end{aligned} \tag{11}$$

with the boundary conditions (BC):

$$\begin{aligned} E_{1,\parallel}|_{z=0} &= E_{2,\parallel}|_{z=0} \Rightarrow E_1 = E_2 \equiv E \\ H_{1,\parallel}|_{z=0} &= H_{2,\parallel}|_{z=0} \Rightarrow H_{1x} = H_{2x} \equiv H_x \\ D_{1,\perp}|_{z=0} &= D_{2,\perp}|_{z=0} \Rightarrow \epsilon E_{1z} = \epsilon E_{2z}. \end{aligned} \tag{12}$$

But, $E_{1z} = E_{2z} = 0$, so the last BC gives us nothing new.

Now, using the curl equation for $\mathbf{E}$,

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \\ &= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right). \end{aligned} \tag{13}$$

So, differentiating and dividing the common exponentials from both sides,

$$\begin{aligned}(+\alpha_1 E, 0, ik_x E) &= \frac{i\xi}{c}\,(H_x, 0, H_{1z}) \text{ for } z > 0 \\ (-\alpha_2 E, 0, ik_x E) &= \frac{i\xi}{c}\,(H_x, 0, H_{2z}) \text{ for } z < 0.\end{aligned} \tag{14}$$

Matching these equations up at the boundary $z = 0$, from the x-component we find that

$$\alpha_1 = -\alpha_2. \tag{15}$$

Since absorption is positive or zero, this could only be satisfied if $\alpha_1 = \alpha_2 = 0$. This is not true for metals.

Strictly speaking we are offering a proof by contradiction. Formally, we make the statement that for a metal, $\alpha_2 \neq 0$ and assume that a TE mode exists. This assumption lead to a violation of $\alpha_2 \neq 0$. So, by contradiction, metals cannot support a TE mode.

4 Problem 3.5 - 10 Points

When a very intense laser irradiates a metal, it ablates material near the surface, resulting in a gas of ionized particles (a plasma) expanding into free space. At which densities of the plasma will light with $\lambda = 1\mu m$ be totally reflected?

4.1 Solution

Reflection occurs when the optical frequency is below the plasma frequency ω_p.

$$\omega_p^2 = \frac{4\pi n e^2}{\epsilon_\infty m_e}. \tag{16}$$

Now, we want $\omega < \omega_p$, so

$$\begin{aligned}\omega^2 &< \frac{4\pi n e^2}{\epsilon_\infty m_e} \\ n &> \frac{\omega^2 \epsilon_\infty m_e}{4\pi e^2}.\end{aligned} \tag{17}$$

Also, $\nu = \frac{c}{\lambda}$, so $\omega = \frac{2\pi c}{\lambda}$. Therefore,

$$\begin{aligned}n &> \left(\frac{2\pi c}{\lambda}\right)^2 \frac{\epsilon_\infty m_e}{4\pi e^2} \\ &> \frac{\pi m_e c^2}{e^2 \lambda^2} \text{ for } \epsilon_\infty \simeq 1.\end{aligned} \tag{18}$$

Remember that this is in Gaussian (cgs) units, so

$$\begin{aligned} &1\text{C} = 3 \times 10^{9}\text{esu} \\ &e = 4.81 \times 10^{-10}\text{esu} \\ &m_e = 9.11 \times 10^{-28}\text{g} \\ &\lambda = 10^{-4}\text{cm} \\ &c = 2.9979 \times 10^{10}\text{cm/s.} \end{aligned} \tag{19}$$

Therefore,

$$\boxed{n > 1 \times 10^{21} \frac{e^-}{cm^3}.} \tag{20}$$

References

[1] Nasser N. Peyghambarian, Stephan W. Koch, and Andre Mysyrowicz. *Introduction to Semiconductor Optics.* Prentice Hall, Englewood Cliffs, NJ, 1993.