Axial irradiance of a focused beam

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The principal maximum of axial irradiance of a focused beam with a low Fresnel number does not lie at its focal point; instead it lies at a point that is closer to the focusing pupil. It has been shown by the numerical example of a weakly truncated Gaussian beam that its value increases and its location moves closer to the pupil when spherical aberration is introduced into the beam. Such an increase has been referred to as "beyond the conventional diffraction limit." Similarly, an increase in the value and a shift in the location of the principal maximum of axial irradiance of a uniform beam toward the pupil by the introduction of some spherical aberration has been characterized as an unexpected result. We explain why and how such a result comes about and that it neither invalidates any diffraction limit nor is it unexpected. We illustrate this for uniform as well as Gaussian beams of various truncation ratios. Both focused and collimated beams aberrated by spherical aberration or astigmatism are considered. © 2005 Optical Society of America

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1. INTRODUCTION

The peak or the maximum of axial irradiance of a weakly truncated focused Gaussian beam with a low Fresnel number lies at its beam waist and not at its focal point.[1–6] Its value can be substantially higher than the focal-point irradiance and its location can be quite far from the focus toward the focusing pupil. Yoshida and Asakura have shown that the peak moves closer to the pupil and its value increases when some negative spherical aberration is introduced into the beam.[7] They expected the axial irradiance at any point to be below its value at the beam waist in the absence of aberration, and thus attributed the increase to being "beyond the conventional diffraction limit." Jiang and Stamnes considered a uniform focused beam and showed by way of a numerical example that the principal maximum of axial irradiance shifts toward the pupil, and its value increases when some spherical aberration is introduced into the beam.[8] They characterized the result as "quite unexpected." We explain that these results are neither beyond the diffraction limit nor surprising.

It is well known that aberrations reduce the focal-point irradiance of a focused beam or the central value of an aberration-free point-spread function (PSF).[9,10] For small aberrations the relative decrease is given (approximately) by the variance of the phase aberration. For example, Rayleigh showed that a quarter wave ($\lambda/4$) of spherical aberration reduces the central irradiance of an aberration-free PSF by 20%, or that the Strehl ratio of the image is 0.80.[11] However, in this case, the central value in a defocused image plane corresponding to a defocus aberration of $-\lambda/4$ is 0.98.[12–14] The Fresnel number of the focusing pupil is assumed to be large so that the change in the distance of the image plane due to defocus is too (negligibly) small to affect the irradiance owing to the inverse-square-law dependence on the distance. Thus by balancing the spherical aberration with an equal but opposite amount of defocus aberration, the central irradiance is increased substantially. The standard deviation of the aberration is reduced by a factor of four from $\lambda/13.42$ to $\lambda/53.67$.

Just as the Strehl ratio of a beam aberrated by a small amount of spherical aberration can be increased if an appropriate amount of defocus aberration is introduced (by observing the beam in a defocused plane), similarly the central irradiance of a focused beam observed in a defocused plane can be increased if an appropriate amount of spherical aberration is introduced into the beam. Accordingly the value and the location of the principal maximum of axial irradiance *can* change by the introduction of spherical aberration. "Diffraction limit" simply implies that the focal-point irradiance of a focused beam is maximum when it is aberration free. However, Yoshida and Asakura have taken it to mean that the axial irradiance at another point (beam waist location in their example) is also maximum when the beam is (otherwise) aberration-free. This, of course, is not true, as demonstrated by their example of a weakly truncated Gaussian beam.[7] A focused beam is referred to as being aberration free if its wavefront at the pupil is spherical with its center of curvature at the focal point. Accordingly, a beam when observed in a plane other than the focal plane is not aberration free; it is in fact aberrated by the defocus aberration. For it to be aberration free, the wavefront must be spherical with its center of curvature at the observation point. In other words, it must be focused at the observation point. When spherical aberration whose amount depends on the location of the defocused point is introduced into the beam, it reduces the variance of the defocus aberration, thereby increasing the irradiance at the defocused point. Accordingly, the results of Yoshida and Asakura and Jiang and Stamnes are to be expected. If a fixed amount of spherical aberration is introduced, it may decrease or increase the aberration variance depending on the location of the defocused point. However, there is nothing unique about spherical aberration in increasing the value and shifting

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the location of the principal maximum of the axial irradiance. Any aberration, e.g., astigmatism, that reduces the variance of the defocus aberration at a certain defocused point will yield a higher irradiance at that point, provided the variance is not too large to invalidate improvement in Strehl ratio due to variance reduction.[6,12–14]

In this paper, we consider uniform as well as Gaussian focused beams and show how spherical aberration or astigmatism affects their axial irradiance. The axial irradiance at a point is determined by two competing factors, namely, the defocus aberration and the inverse-square-law dependence on the distance of the point from the plane of the focusing pupil. The defocus aberration reduces the central irradiance in an observation plane and the inverse-square-law dependence increases it for points closer to the pupil (than the focal point) and decreases it for those that are farther. Hence the irradiance at axial points closer to the pupil can be higher than the focal-point irradiance if the increase due to the inverse-square-law dependence is higher than the decrease due to the defocus aberration. This happens in systems with small Fresnel numbers, since they have a large depth of focus.[15] As a result, the observation distance can change significantly without introducing significant defocus aberration. Accordingly the axial irradiance for such systems is highly asymmetric about the focal point and its principal maximum lies at a point that is closer to the pupil. Of course, the 2D distributions in orthogonal planes are also asymmetric about the focal plane.[16] Similarly, when a certain amount of spherical aberration is introduced into the beam such that it balances the defocus aberration by reducing its aberration variance across the pupil for some region of the axial distance, the degrading effect of the defocus aberration is reduced, yielding a higher axial irradiance in that region. Systems with large Fresnel numbers have a small depth of focus. Accordingly, the effect of the inverse-square-law dependence is negligible, and the axial irradiance away from the focus decreases as a result of the defocus aberration. The axial irradiance (and the 2D distributions in orthogonal planes) for such systems in the vicinity of the focal point (or focal plane) is symmetric about it. We illustrate these results by considering numerical examples of Gaussian beams with a unity Fresnel number, but varying truncation ratio. We show that the axial irradiance of a beam aberrated by an appropriate amount of spherical aberration or astigmatism is higher in the vicinity of focus closer to the focusing pupil than that for an (otherwise) aberration-free beam. A uniform beam is treated as a limiting case of a Gaussian beam and a collimated beam is treated as a limiting case of a focused beam.

2. THEORY

A. Pupil Function

Consider a Gaussian beam of wavelength λ whose amplitude distribution at the exit pupil of a system is given by

$$A(\rho) = A_0 \exp(-\gamma\rho^2), \qquad 0 \leq \rho \leq 1, \tag{1}$$

where A_0 is a constant (with A_0^2 having dimensions of $\mathrm{W/m^2}$), ρ is the distance of a point in the plane of the pupil from its center normalized by its radius a, and

$$\gamma = (a/\omega)^2 \tag{2}$$

is the truncation parameter of the beam with ω as the Gaussian radius representing the radial distance at which the amplitude decrease to $1/e$ of its value at the center. For a total power P transmitted by a pupil of area $S_p = \pi a^2$, A_0^2 is given by

$$A_0^2 = \frac{2\gamma}{1-\exp(-2\gamma)}\left(\frac{P}{S_p}\right). \tag{3}$$

The case of a uniform pupil is obtained by letting $\gamma \to 0$. A large value of γ or $\omega \ll a$ represents a narrow Gaussian or a weakly truncated beam.

B. Focused Beam

The axial irradiance at a distance z of a beam focused at a distance R is given by[17]

$$I(z;\gamma) = \left(\frac{R}{z}\right)^2 \left(\frac{2\gamma}{B_d^2+\gamma^2}\right)\frac{1}{\sinh\gamma}(\cosh\gamma - \cos B_d), \tag{4}$$

where

$$B_d(z) = \pi N(R/z - 1) \tag{5}$$

is the peak value of the defocus phase aberration of a spherical wavefront of radius of curvature R with respect to a reference sphere of radius of curvature z. Here $N = a^2/\lambda R$ is the Fresnel number representing the number of Fresnel zones in the pupil as observed from the focus. The irradiance is in units of the focal-point irradiance PS_p/λ^2R^2 for a uniform beam, where P is the power transmitted by a pupil of area S_p. The axial irradiance goes through a series of maxima and minima as a function of z because of the $\cos B_d$ term. The focal-point irradiance is given by

$$I(R;\gamma) = [\tanh(\gamma/2)]/(\gamma/2). \tag{6}$$

Its value is ≤ 1, equality holding for $\gamma = 0$, illustrating that maximum central irradiance is obtained for a uniform pupil.[10,18] By equating to zero the derivative of axial irradiance with respect to z, we obtain the positions of its maxima and minima as the solutions of

$$2\left(\frac{\lambda z}{S_p} - \frac{B_d}{B_d^2+\gamma^2}\right)(\cosh\gamma - \cos B_d) = -\sin B_d. \tag{7}$$

They occur approximately at those z values at which the pupil subtends an odd or an even number of Fresnel zones, respectively. Since $\cosh\gamma > 1$ and $\cos B_d \leq 1$, the minima are not equal to zero (unless $\gamma = 0$, representing a uniform beam). For a given value of N, the location of the principal maximum moves closer to the pupil as γ increases.

The axial irradiance given by Eq. (4) results from two competing factors. The inverse-square-law dependence on the distance z yields the $(R/z)^2$ factor. The rest of the expression on the right-hand side represents the effect of defocus as an aberration. The ratio of this second factor and the focal-point irradiance represents the Strehl ratio for the defocus aberration, i.e., it represents the ratio of axial irradiance at a distance z when the beam is focused at a distance R to that when it is focused at a distance z.[19]

This ratio is less than one, since an aberration always reduces the central irradiance (from its aberration-free value). The axial irradiance at a certain distance $z<R$ is larger than the focal-point irradiance if the increase due to the inverse-square law is larger than the decrease due to the defocus aberration.

For large values of γ, i.e., for a weakly truncated beam, Eqs. (4) and (6) reduce to

$$I(z;\gamma) = \left(\frac{R}{z}\right)^2 \frac{2\gamma}{B_d^2 + \gamma^2}, \tag{8}$$

$$I(R) = 2/\gamma, \tag{9}$$

respectively. In this case, practically all of the power is transmitted by the pupil and a Gaussian beam remains Gaussian as it propagates.[20–22] Only a fraction $\exp(-2\gamma)$ is not transmitted. As $z \rightarrow 0$, Eq. (8) yields

$$I(0;\gamma) = 2\gamma/\pi^2 N^2. \tag{10}$$

Multiplying by the normalization factor $PS_p/\lambda^2 R^2$, this value is practically equal to the central pupil irradiance A_0^2, as expected. The maximum or the peak value of axial irradiance is given by

$$I(z_p;\gamma) = (2/\gamma) + (2\gamma/\pi^2 N^2), \tag{11}$$

which occurs at the beam waist at a distance z_p given by

$$z_p/R = [1 + (\gamma/\pi N)^2]^{-1}. \tag{12}$$

The peak irradiance is the sum of the focal-point irradiance and pupil irradiance. Substituting Eq. (12) into Eq. (5), we note that the location z_p corresponds to a defocus aberration of $\gamma^2/\pi N$. It is evident from Eqs. (8) and (9) that this defocus aberration yields a Strehl ratio

$$S = [1 + (B_d/\gamma)^2]^{-1}. \tag{13}$$

Its value at z_p for $N=1$ is 0.1086.

For a uniform beam, i.e., for $\gamma=0$, Eqs. (4) and (7) reduce to

$$I(z) = (R/z)^2 \{[\sin(B_d/2)]/(B_d/2)\}^2, \tag{14}$$

$$\tan(B_d/2) = (R/z) B_d/2, \qquad z \neq R, \tag{15}$$

respectively. The focal-point irradiance $I(R)$ is unity, since it was used to normalize the irradiance in Eq. (4).

The axial irradiance of a beam aberrated by aberration $\Phi(\rho,\theta)$ is given by[23]

$$I(z;\gamma) = \frac{2\gamma}{1-\exp(-2\gamma)} \left(\frac{R}{\pi z}\right)^2 \left| \int_0^1 \int_0^{2\pi} \exp(-\gamma\rho^2) \times \exp\{i[\Phi(\rho,\theta) + B_d\rho^2]\} \rho \mathrm{d}\rho \mathrm{d}\theta \right|^2. \tag{16}$$

For large values of γ, $\exp(-2\gamma)$ may be neglected compared to unity and the upper limit on the radial integration may be replaced by infinity with negligible error. For a beam aberrated by spherical aberration $A_s\rho^4$, Eq. (16) reduces to

$$I(z;\gamma) = \frac{2\gamma}{1-\exp(-2\gamma)} \left(\frac{R}{z}\right)^2 \times \left| \int_0^1 \exp(-\gamma x) \exp[i(A_s x^2 + B_d x)] \mathrm{d}x \right|^2. \tag{17}$$

Similarly, for a beam aberrated by astigmatism $A_a\rho^2\cos^2\theta$, it reduces to

$$I(z;\gamma) = \frac{2\gamma}{1-\exp(-2\gamma)} \left(\frac{R}{2}\right)^2 \left| \int_0^1 \exp(-\gamma x) \times \exp[i(0.5A_a + B_d)x] J_0(0.5A_a x) \mathrm{d}x \right|^2, \tag{18}$$

where we have used the fact that

$$\begin{aligned} \int_0^{2\pi} \exp(iA_a\rho^2\cos^2\theta)\mathrm{d}\theta &= \exp(0.5iA_a\rho^2) \int_0^{2\pi} \exp(0.5iA_a\rho^2\cos 2\theta)\mathrm{d}\theta \\ &= 2\pi \exp(0.5iA_a\rho^2) J_0(0.5A_a\rho^2). \end{aligned} \tag{19}$$

C. Balancing of Defocus Aberration with Spherical Aberration or Astigmatism

The variance of an aberration $\Phi(\rho,\theta)$ balanced by defocus aberration $B_d\rho^2$ across the Gaussian weighted pupil is given by

$$\sigma_\Phi^2 = \langle \Phi^2 \rangle - \langle \Phi \rangle^2, \tag{20}$$

where

$$\begin{aligned} \langle \Phi^n \rangle &= \int_0^1 \int_0^{2\pi} A(\rho)[\Phi(\rho,\theta) + B_d\rho^2]^n \rho \mathrm{d}\rho \mathrm{d}\theta \Big/ \int_0^1 \int_0^{2\pi} A(\rho) \rho \mathrm{d}\rho \mathrm{d}\theta \\ &= \{\gamma/\pi[1-\exp(-\gamma)]\} \int_0^1 \int_0^{2\pi} \exp(-\gamma\rho^2)[\Phi(\rho,\theta) + B_d\rho^2]^n \rho \mathrm{d}\rho \mathrm{d}\theta, \end{aligned} \tag{21}$$

with $n=1$ or 2. Generally we are interested in balancing spherical aberration or astigmatism of a beam with defocus aberration to minimize its variance and thereby reduce its degrading effect.[6,9] However, here we are interested in balancing defocus aberration with spherical aberration or astigmatism to reduce its variance and thereby increase the axial irradiance at points closer to the pupil. Table 1 lists the standard deviation of defocus aberration with and without balancing with spherical aberration or astigmatism, showing that it decreases as γ increases. Spherical aberration reduces the standard deviation by a factor of 4, 3.74, and $\sqrt{5}$ when $\sqrt{\gamma}=0$, 1, and 3, respectively. The corresponding reduction factors for astigmatism are $\sqrt{3/2}$, 1.27, and $\sqrt{2}$. The reduction factor decreases with increasing γ in the case of spherical aberration, but increases in the case of astigmatism. The reduction factor is the same whether spherical aberration is balanced with defocus aberration or defocus aberration is balanced with spherical aberration, but the balanced aberrations have different forms in the two cases. For example, for a uniform beam, spherical aberration $A_s\rho^4$ balanced with defocus aberration is $A_s(\rho^4-\rho^2)$, but defocus aberration $B_d\rho^2$ balanced with spherical aberration is $B_d[\rho^2-(15/16)\rho^4]$. Similarly, astigmatism $A_a\rho^2\cos^2\theta$ balanced with defocus aberration is $A_a(\rho^2\cos^2\theta-\rho^2)$, but defocus aberration balanced with astigmatism is $B_d[\rho^2-(2/3)\rho^2\cos^2\theta]$. The difference comes from the fact that optimal balancing is done with respect to defocus aberration in one case and spherical aberration or astigmatism in the other.

It should be noted that minimum variance yields maximum central irradiance only for small aberrations. For large aberrations, nonoptimal balancing yields maximum central irradiance. This may be seen by considering, for example, a certain value of z that corresponds to a certain value of B_d, and determining the value of A_s that yields maximum central irradiance. Thus we consider the Strehl ratio of a focused beam representing the ratio of the axial irradiance at a distance z with and without spherical aberration:

$$S=\left[\frac{\gamma}{1-\exp(-\gamma)}\right]^2\left|\int_0^1 \exp(-\gamma x)\exp[i(B_d x+A_s x^2)]\mathrm{d}x\right|^2, \tag{22}$$

and determine the value of A_s that maximizes the irradiance.

D. Collimated Beam

A collimated beam is equivalent to a beam focused at infinity or one with a Fresnel number of zero. Thus letting $R\to\infty$ in Eqs. (5) and (16), we obtain

$$I(z;\gamma)=\frac{2\gamma(B_d/\pi)^2}{1-\exp(-2\gamma)}\left|\int_0^1\int_0^{2\pi}\exp(-\gamma\rho^2)\times\exp\{i[\Phi(\rho,\theta)+B_d\rho^2]\}\rho\mathrm{d}\rho\mathrm{d}\theta\right|^2, \tag{23}$$

where

$$B_d=\pi/4z \tag{24}$$

represents the peak value of the defocus phase aberration of a plane wave front with respect to a reference sphere of radius of curvature z. Now the irradiance is in units of the pupil irradiance P/S_p for a uniform pupil and z is in units of the far-field distance D^2/λ, where $D=2a$ is the pupil diameter. When $\Phi(\rho,\theta)=0$, Eq. (23) reduces to

$$I(z;\gamma)=\{2\gamma/[1+(4\gamma z/\pi)^2]\}[\coth\gamma-\cos(\pi/4z)/\sinh\gamma]. \tag{25}$$

For a uniform or a weakly truncated Gaussian beam, Eq. (25) further reduces to

Table 1. Standard Deviation of Defocus Aberration and Defocus Aberration Balanced[a] with Spherical Aberration or Astigmatism for Minimum Variance

	Standard Deviation		
Aberration, $B_d\rho^2+\Phi(\rho,\theta)$	$\gamma=0$	$\gamma=1$	$\sqrt{\gamma}\geqslant 3$
Defocus, $B_d\rho^2$	$\frac{B_d}{2\sqrt{3}}=\frac{B_d}{3.46}$	$\frac{B_d}{3.55}$	$\frac{B_d}{\gamma}$
Defocus aberration balanced with spherical aberration, $B_d\rho^2+A_s\rho^4$	$\frac{B_d}{8\sqrt{3}}=\frac{B_d}{13.86}$	$\frac{B_d}{13.27}$	$\frac{B_d}{\sqrt{5}\gamma}$
	$\left(A_s=-\frac{15}{16}B_d\right)$	$(A_s=-B_d)$	$\left(A_s=-\frac{\gamma}{5}B_d\right)$
Defocus aberration balanced with astigmatism, $B_d\rho^2+A_a\rho^2\cos^2\theta$	$\frac{B_d}{3\sqrt{2}}=\frac{B_d}{4.24}$	$\frac{B_d}{4.53}$	$\frac{B_d}{\sqrt{2}\gamma}$
	$\left(A_a=-\frac{2}{3}B_d\right)$	$(A_a=-0.77B_d)$	$(A_a=-B_d)$

[a]The amount of the balancing aberration is listed for each case in parentheses.

$$I(z;0) = 4\sin^2(\pi/8z) \tag{26}$$

or

$$I(z;\gamma) = 2\gamma/[1 + (4\gamma z/\pi)^2], \tag{27}$$

as may also be seen from Eqs. (14) and (8), respectively.

The defocus aberration can be balanced with spherical aberration or astigmatism in exactly the same manner as for a focused beam. Thus the results of Table 1 apply equally well for a collimated beam. The only significant difference is in the definition of the defocus coefficient B_d. Hence the axial irradiance of a beam optimally balanced with spherical aberration or astigmatism (whose amount varies with z) is given by

$$I(z;\gamma) = \frac{2\gamma B_d^2}{1 - \exp(-2\gamma)} \left| \int_0^1 \exp(-\gamma x)\exp[i(A_s x^2 + B_d x)]\mathrm{d}x \right|^2, \tag{28}$$

$$I(z;\gamma) = \frac{2\gamma B_d^2}{1 - \exp(-2\gamma)} \left| \int_0^1 \exp(-\gamma x) \times \exp[i(0.5A_a + B_d)x]J_0(0.5A_a x)\mathrm{d}x \right|^2, \tag{29}$$

respectively.

3. NUMERICAL RESULTS

Figure 1 shows the axial irradiance of a Gaussian beam focused at a distance R with $N=1$ in units of the focal-point irradiance PS_p/λ^2R^2 for a uniform beam, i.e., for $\gamma=0$. The beam is aberration-free except for the defocus aberration resulting from the observation in a defocused plane. The focal-point irradiance when $\gamma=1$, 4, and 9 is equal to 0.924, 0.482, and 0.222, respectively. For a uniform beam, the principal maximum lies at $z=0.6R$ with a value of 1.89 (compared with a value of unity at the focal point). The minima have a value of zero at z values given by $z/R=1/3, 1/5, 1/7$, etc., as may be seen from Eqs. (5) and (14). Similar results are obtained for a Gaussian beam with $\gamma=1$, except that the minima have a nonzero value as a result of the incomplete cancellation of the even number of Fresnel zones subtended at these locations owing to their different amplitudes. For large values of γ, the secondary maxima and minima disappear, as illustrated for $\sqrt{\gamma}=3$. The value at $z=0$ is given by $2\gamma/\pi^2$, or equal to 1.824. The peak in this case is located at the beam waist at $z=0.1086R$ with a value of 2.046, as may be seen from Eqs. (10) and (11). The peak value is 9.2 times the corresponding focal-point irradiance of 0.222 and 1.12 times the central pupil irradiance of 1.824. The defocus aberration of $B_d=81/\pi$ or 4.10λ reduces the irradiance by a factor of 0.1086, but the inverse-square law increases it by a factor of 84.79.

Figure 2 shows how spherical aberration modifies the axial irradiance of a beam. The variation of $|B_d|$ with z is also shown in this figure. The value of B_d is positive for $z/R<1$ and negative for $z/R>1$. The solid curve in Fig. 2(a) shows the axial irradiance of a uniform beam aberrated by spherical aberration $A_s=-5\pi/8$ or $-(5/16)\lambda$ that minimizes the variance of defocus aberration of $\lambda/3$ corresponding to the z value where the principal maximum lies in the absence of spherical aberration. Compared to when $A_s=0$, the irradiance is smaller in the vicinity of the focal point and larger in the vicinity of the principal maximum. The peak moves closer to the pupil, the secondary maxima are higher, and the minima are no longer zero. If the amount of spherical aberration varies as B_d varies with z/R so that $A_s=-(15/16)B_d$, then the aberration variance is minimum and the axial irradiance is higher for any value of z (although the difference in the vicinity of the focal point is negligible), as illustrated by the dashed curve. The peak value now lies even closer to the pupil, and is more than ten times the focal-point irradiance. Thus the peak value with spherical aberration is more than five times the peak value without it. Figure 2(b) shows similar results for a Gaussian beam with $\gamma=1$. Minimum variance of the defocus aberration when balanced with spherical aberration is obtained in this case when $A_s=-B_d$. (A more precise relationship is $A_s=-0.995B_d$.) The maxima for a Gaussian beam are lower compared with those for a uniform beam.

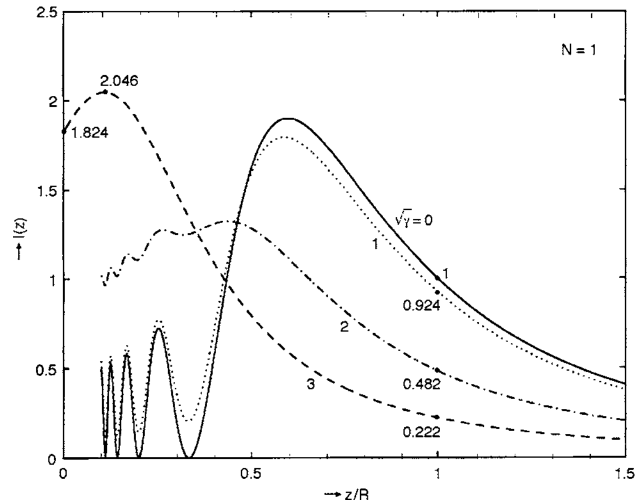

Fig. 1. Axial irradiance of a truncated Gaussian beam focused at a distance R with a Fresnel number $N=1$. The truncation parameter $\sqrt{\gamma}=a/\omega$, where a is the radius of the pupil truncating a beam of radius ω representing the beam radius at which the amplitude reduces to $1/e$ of its value at the center. $\gamma=0$ corresponds to a uniform beam and $\sqrt{\gamma}=3$ represents a weakly truncated beam. The irradiance is normalized by the focal-point irradiance PS_p/λ^2R^2 for a uniform beam, where P is the power transmitted by a pupil of area S_p.

As illustrated in Fig. 2(c) for a weakly truncated Gaussian beam with $\sqrt{\gamma}=3$, the peak moves toward the focus and its value more than doubles when spherical aberration $A_s=-729/5\pi$ or -7.39λ is introduced, minimizing the variance of the defocus aberration of 4.10λ corresponding to the peak location. Aberration balancing reduces the standard deviation of the defocus aberration by a factor of $\sqrt{5}$ from a value of B_d/γ to $B_d/\sqrt{5}\gamma$. If the amount of spherical aberration varies as B_d varies with z/R so that $A_s=-(9/5)B_d$, then the peak location does not change but its value increases significantly. A peak with a fourfold in-

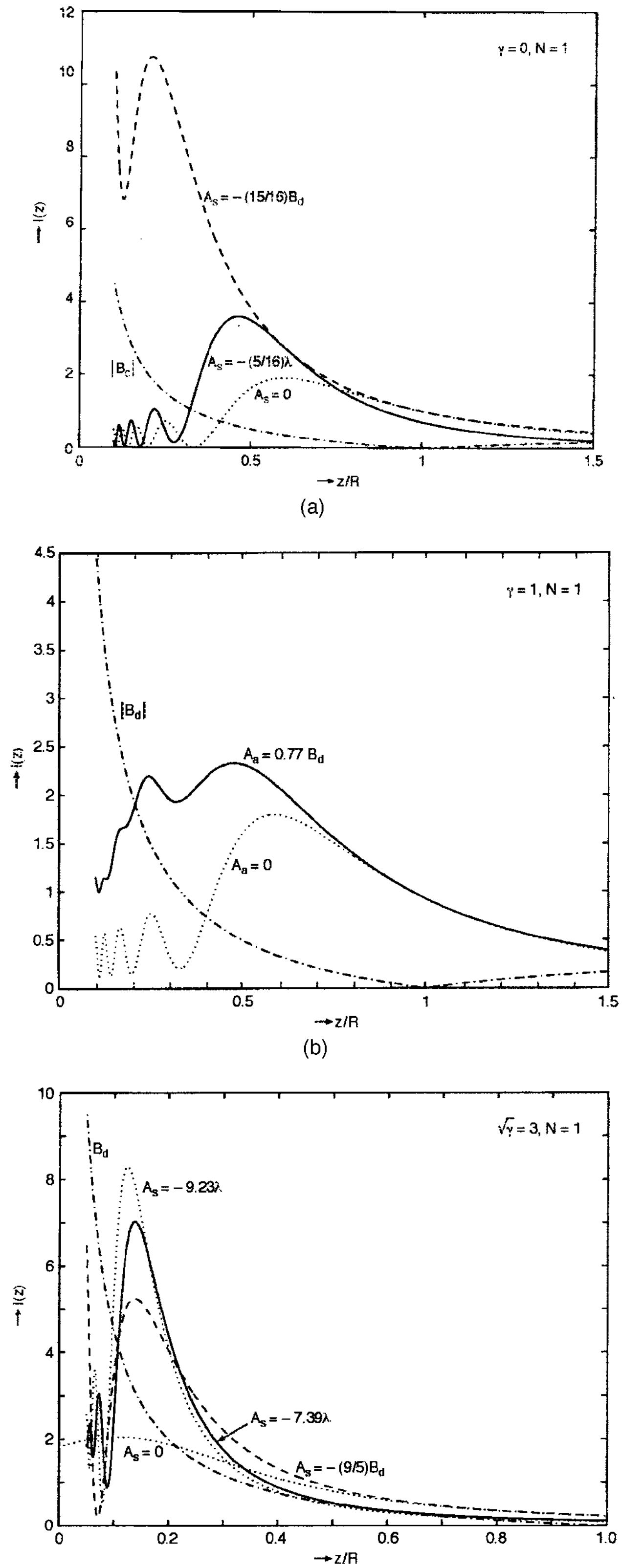

Fig. 2. Axial irradiance of a focused beam with a Fresnel number $N=1$ aberrated by spherical aberration A_s. The axial irradiance when $A_s=0$ is shown for comparison. The defocus aberration $|B_d|$ in units of wavelength is also shown. (a) Uniform beam ($\gamma=0$). (b) Gaussian beam with $\gamma=1$. (c) Weakly truncated Gaussian beam with $\sqrt{\gamma}=3$.

crease that is slightly closer to the pupil is obtained when $A_s=-9.23\lambda$.

Figure 3 shows how the axial irradiance is modified when defocus aberration is balanced with astigmatism. The increase is not as dramatic as in the case of spherical aberration when $\gamma=0$ or 1. This is consistent with the fact that aberration balancing also in this case does not reduce the standard deviation of the defocus aberration significantly. However, the axial irradiance of a weakly truncated beam with $\sqrt{\gamma}=3$ increases significantly and monotonically as z decreases.

Figure 4 shows that the Strehl ratio for a given value of defocus aberration B_d varies as a function of spherical aberration A_s. Both B_d and A_s are in units of wavelength λ. It is evident that, as the amount of defocus aberration increases, the maximum Strehl ratio is obtained for an A_s value that is different from the one yielding minimum variance. For example, when $B_d=3\lambda$, maximum Strehl ratio is obtained when $A_s=-2.2\lambda$ instead of -2.8λ when $\gamma=0$, and $A_s=-2\lambda$ instead of -3λ when $\gamma=1$. When $\sqrt{\gamma}=3$, the Strehl ratio peaks to the right of the expected value of $A_s=-(9/5)B_d$.

Figure 5 illustrates how the axial irradiance of aberration-free, collimated uniform and Gaussian beams varies with distance z from the pupil. The irradiance in this figure is in units of the pupil irradiance P/S_p for a uniform pupil and z is in units of the far-field distance D^2/λ. Compared with Fig. 1, this figure corresponds to $N=0$. The axial irradiance of a uniform beam goes through a series of maxima with a value of 4 and minima of zero value until it decreases with distance approximately according to the inverse-square law. The maxima in the case of a Gaussian beam with $\gamma=1$ are somewhat higher and the minima have a nonzero value. Because of their different amplitudes, the Fresnel zones in this case do not cancel completely at the location of the minima. For large values of γ, e.g., $\sqrt{\gamma}=3$, the axial irradiance peaks at the pupil and decreases monotonically as z increases.

Figure 6 illustrates the effect of spherical aberration on the axial irradiance of a collimated beam. The aberration-free axial irradiance is included in this figure for comparison. The solid curves show that the axial irradiance increases dramatically near the pupil, especially for small values of γ, as a varying amount of spherical aberration minimizing the variance of the defocus aberration is introduced. Similar results are obtained when astigmatism is introduced, as illustrated in Fig. 7, though the increase at axial points near the pupil is not as dramatic. Of course, the irradiance at a certain distance in the absence of an aberration is higher if the beam is focused at that distance. For example, the irradiance of a uniform beam focused at a distance z is $(\pi/4z)^2$. Its value at $z=0.1$, corresponding to $B_d=1.25\lambda$, is 62 compared to a value of 2 or 44 for a collimated beam with $A_s=0$ or $A_s=-(15/16)B_d$, respectively.

4. CONCLUSIONS

The peak or the principal maximum of axial irradiance of a focused Gaussian beam with a low Fresnel number does not lie at its focal point. Its value can be substantially

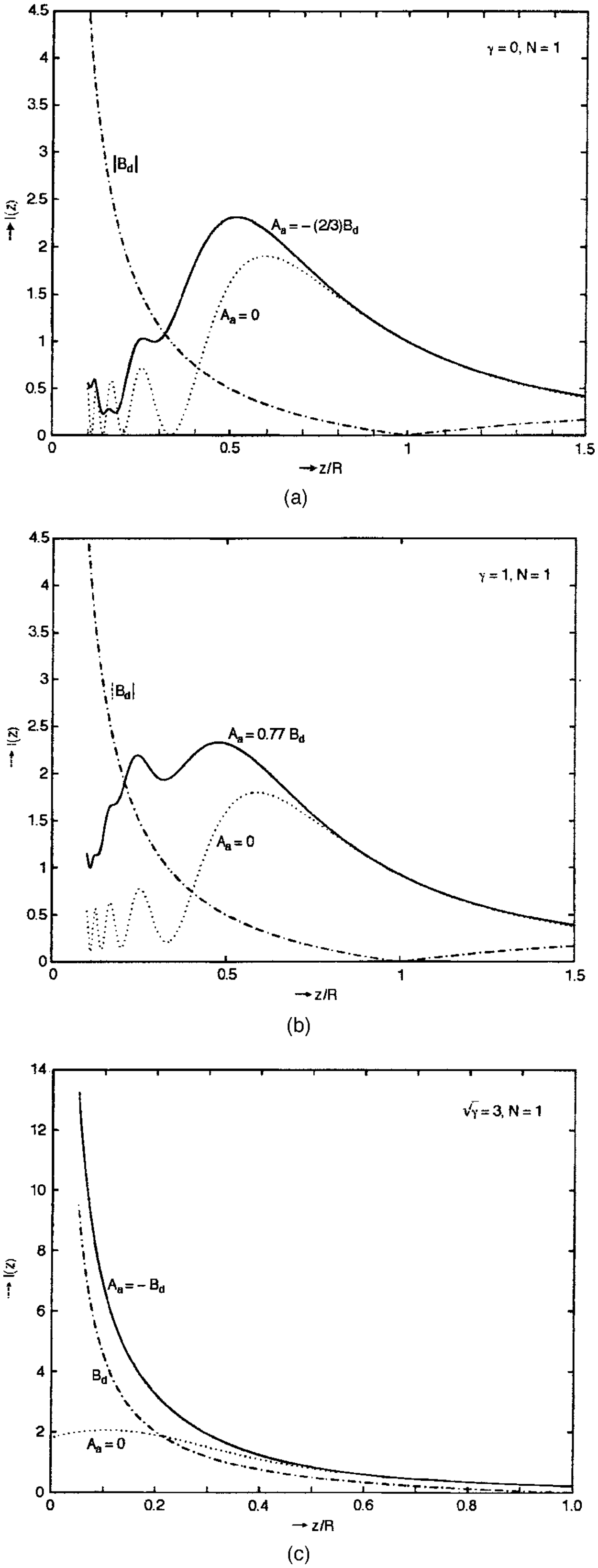

Fig. 3. Axial irradiance of a focused beam with a Fresnel number $N=1$ aberrated by astigmatism A_a. The axial irradiance when $A_a=0$ is shown for comparison. The defocus aberration $|B_d|$ in units of wavelength is also shown. (a) Uniform beam ($\gamma=0$). (b) Gaussian beam with $\gamma=1$. (c) Weakly truncated Gaussian beam with $\sqrt{\gamma}=3$.

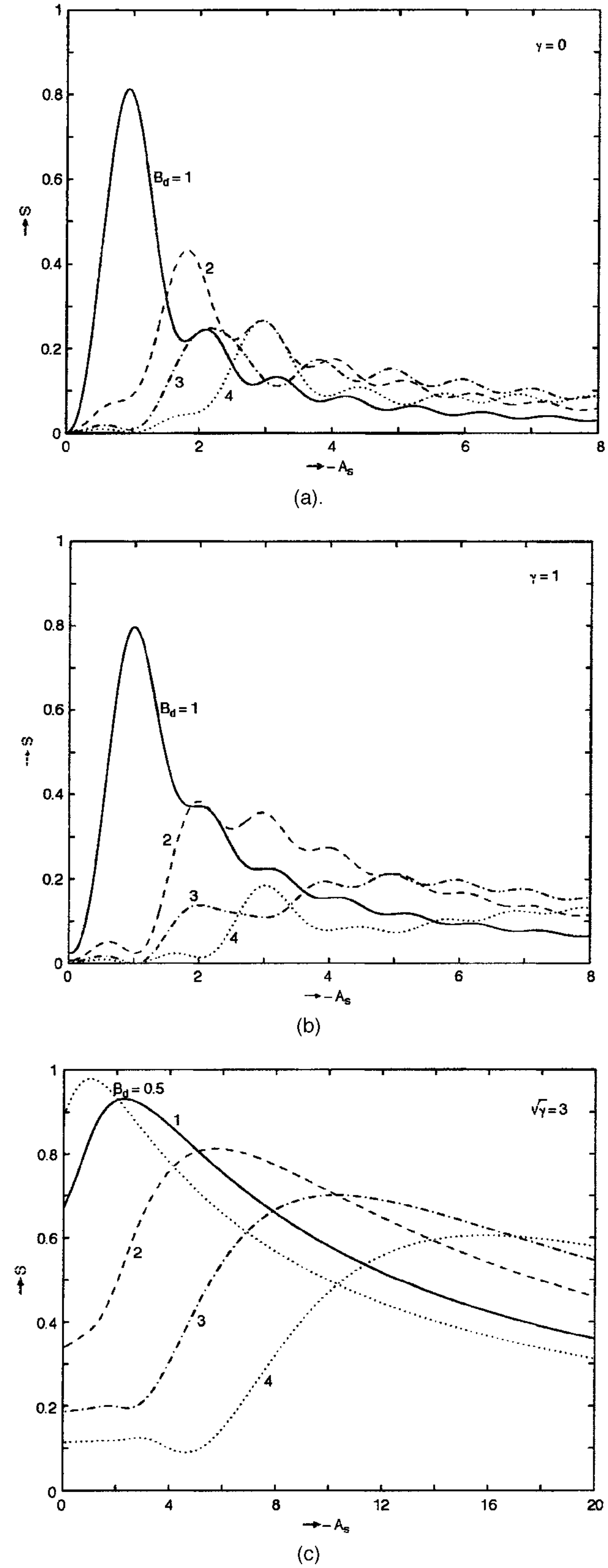

Fig. 4. Strehl ratio of a beam for a given value of defocus aberration B_d as a function of spherical aberration A_s. Both B_d and A_s are in units of wavelength λ. (a) Uniform beam ($\gamma=0$); the Strehl ratio in this case is zero when B_d is an integral number of wavelengths, as may be seen from Eq. (14). (b) Gaussian beam with $\gamma=1$. (c) Weakly truncated Gaussian beam with $\sqrt{\gamma}=3$.

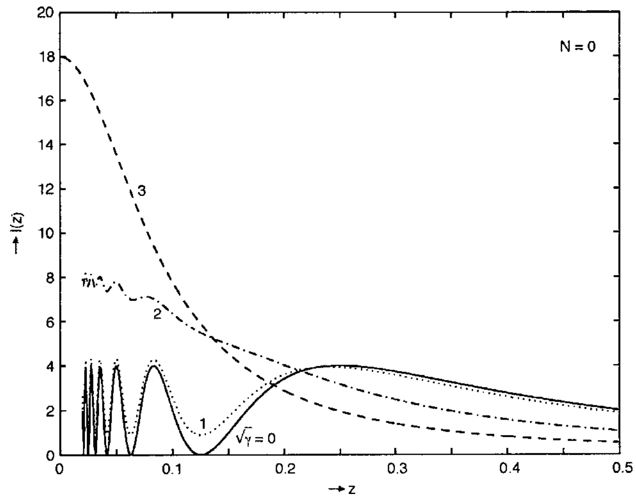

Fig. 5. Axial irradiance of a collimated Gaussian beam, i.e., one with a Fresnel number $N=0$. Uniform ($\gamma=0$) and Gaussian beams with $\sqrt{\gamma}=1,2,3$ are considered. The irradiance is in units of the pupil irradiance P/S_p for a uniform beam and the distance z is in units of the far-field distance D^2/λ.

higher than the focal-point irradiance and its location can be quite far from the focus toward the focusing pupil. By considering a weakly truncated focused Gaussian beam, Yoshida and Asakura showed that the peak moves closer to the pupil and its value increases when some spherical aberration is introduced into the beam.[7] They expected the axial irradiance at any point to be below its value at the beam waist in the absence of aberration. They described this increase as being “beyond the conventional diffraction limit.” We have explained that the result they obtained is neither surprising nor beyond the diffraction limit. “Diffraction limit” simply implies that the focal-point irradiance of a focused beam is maximum when it is aberration-free. However, a focused beam observed in a plane other than the focal plane is not aberration free, since it is in fact aberrated by the defocus aberration. When spherical aberration whose amount depends on the location of the defocused point is introduced into the beam, it balances the defocus aberration and reduces its variance, thereby increasing the irradiance at the defocused point. If a fixed amount of spherical aberration is introduced, it may decrease or increase the aberration variance depending on the location of the defocused point. Hence there is no violation of the diffraction limit. The results of a numerical example of a uniform focused beam considered by Jiang and Stamnes[8] and characterized by them as unexpected can be explained in a similar manner as a consequence of balancing of the defocus aberration by spherical aberration. It should be emphasized that there is nothing unique about spherical aberration’s increasing the value and shifting the location of the principal maximum of the axial irradiance. Any aberration that reduces the variance of the defocus aberration at a certain defocused point will yield a higher irradiance at that point (provided the variance is not too large to invalidate improvement in Strehl ratio as a result of variance reduction[6,12]). We have demonstrated this by introducing astigmatism instead of spherical aberration. Similar, though not as dramatic, results are obtained when astig-

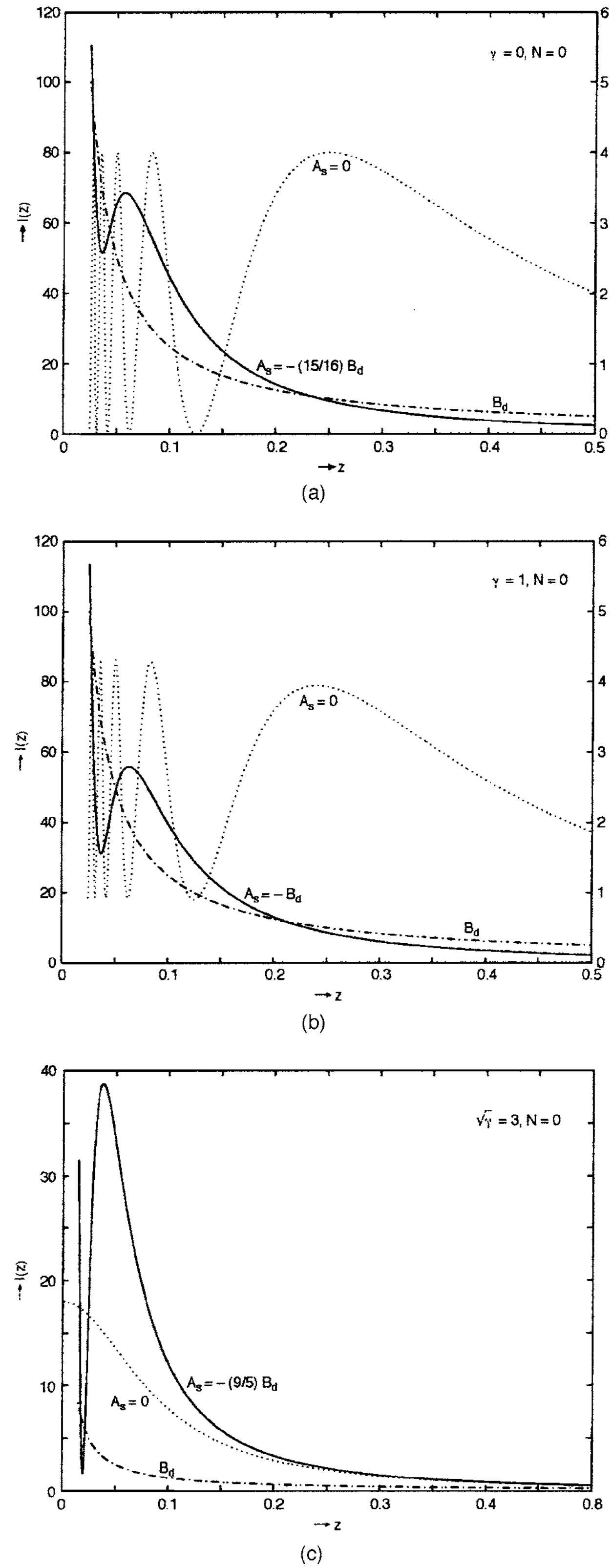

Fig. 6. Axial irradiance of a collimated beam, i.e., one with Fresnel number $N=0$, aberrated by spherical aberration A_s. The axial irradiance when $A_s=0$ is shown for comparison. The units of irradiance and z are the same as in Fig. 5. The defocus aberration B_d in units of wavelength is also shown. (a) Uniform beam ($\gamma=0$). (b) Gaussian beam with $\gamma=1$. (c) Weakly truncated Gaussian beam with $\sqrt{\gamma}=3$. In (a) and (b) the right-hand scale is for $A_s=0$ and B_d.

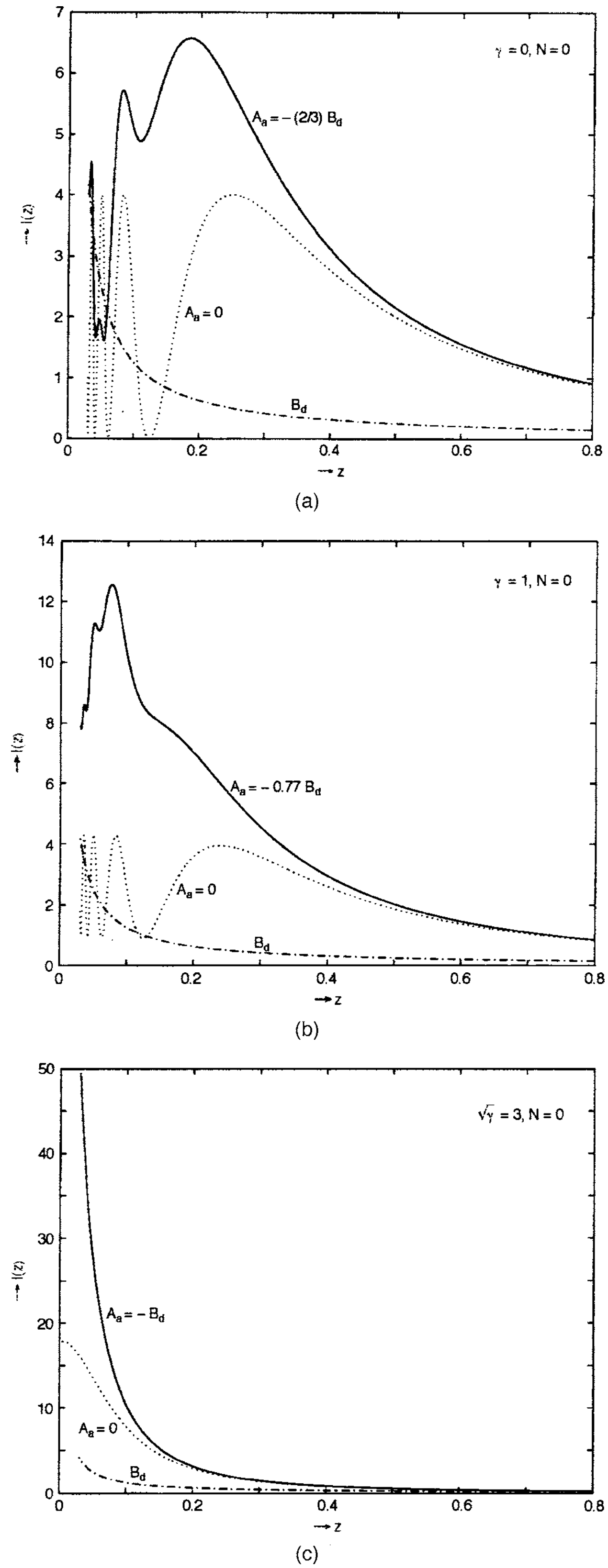

Fig. 7. Axial irradiance of a collimated beam, i.e., one with Fresnel number $N=0$, aberrated by astigmatism A_a. The axial irradiance when $A_a=0$ is shown for comparison. The units of irradiance and z are the same as in Fig. 5. The defocus aberration B_d in units of wavelength is also shown. (a) Uniform beam ($\gamma=0$). (b) Gaussian beam with $\gamma=1$. (c) Weakly truncated Gaussian beam with $\sqrt{\gamma}=3$.

matism is introduced. Such results are independent of the shape of the pupil. They are, for example, equally applicable to systems with annular pupils. The details of aberration balancing and the numerical values of the irradiance and location of the principal maximum will depend on the obscuration ratio of the pupil.

Although the increase in the axial irradiance in the vicinity of the focal point due to the addition of spherical aberration or astigmatism yields a larger depth of focus on the pupil side, maximum irradiance on a target at a certain distance is always obtained when a beam is focused on it, regardless of the fact that the irradiance at points closer to the pupil may be higher.[6,15]

A collimated beam is equivalent to a beam focused at infinity or one with a zero Fresnel number. Except for the definition of the defocused coefficient B_d, balancing of the defocus aberration of a collimated beam with spherical aberration or astigmatism is exactly the same as for a focused beam. Hence it is not surprising that the axial irradiance of a collimated beam is modified in a manner similar to that of a focused beam when spherical aberration or astigmatism is introduced into the beam.

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