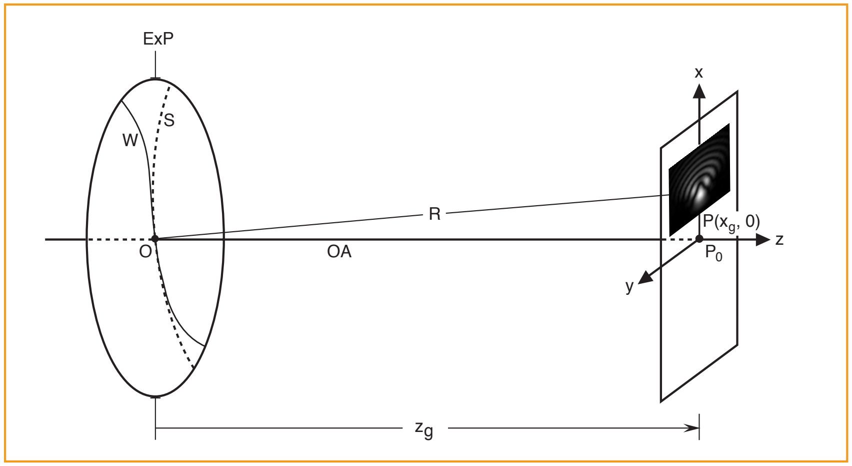
Optical Imaging and Aberrations



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Lecture 1: Rayleigh-Sommerfeld Theory of Diffraction, PSF, and Strehl Ratio

Text Book:

V. N. Mahajan, *Optical Imaging and Aberrations*, Part II: *Wave Diffraction Optics* (2001, SPIE Press).

Reference Books:

M. Born and E. Wolf, *Principles of Optics*, 7th edition (Oxford, New York, 1999), Chapter IX, "The Diffraction Theory of Aberrations," pp. 517-553.

D. J. Schroeder, *Astronomical Optics* (Academic Press, 1987), Chapter 10 "Diffraction Theory and Aberrations" and Chapter 11 "Transfer Functions."

Course Grade:

Home Work	40%
Mid Term Test	30%
Final Test	30%

Lecture 1 Summary

Rayleigh-Sommerfeld Theory of Diffraction, PSF, and Strehl Ratio

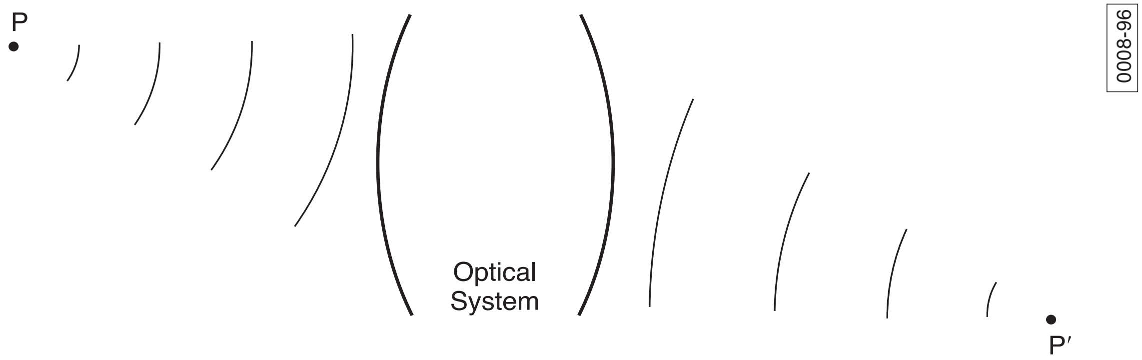
- Rayleigh-Sommerfeld Theory of Diffraction
  - Huygens-Fresnel Principle
  - Fresnel Approximation
    - PSF and OTF
  - Fraunhofer Approximation
    - PSF and OTF

- Foundations of Image Formation
  - Gaussian Image
  - Diffraction Image
    - Shift-Invariant Imaging

- Point-Spread Function

- Strehl Ratio

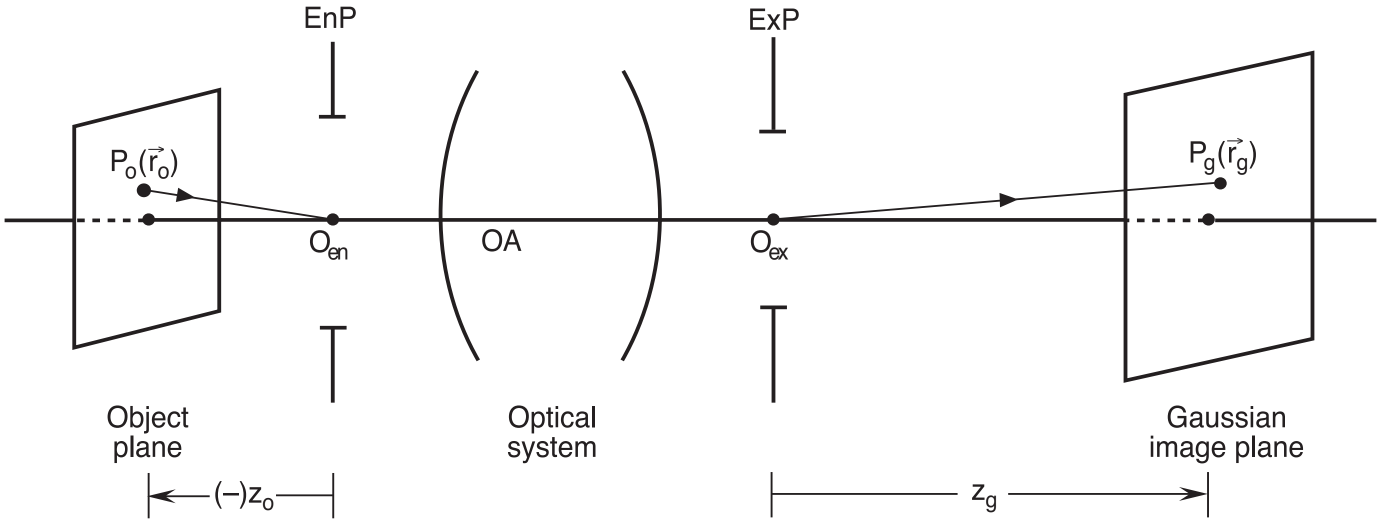
Optical Imaging Fundamentals



- A spherical wave diverging from a point object P is converted by a perfect imaging system into a spherical wave converging to the image point P'. The optical path lengths of the rays from P to P' are equal. The location of the image point is determined by use of Gaussian optics.

- In reality, however, the image formed is not a point. It consists of a bright spot surrounded by dark and bright rings, called the *Airy pattern*, and the image is said to be aberration free.

- In practice, the imaging system does not convert the diverging spherical wave into a perfect spherical wave converging to the image point, and the optical path lengths of the rays from the object point P to the image point P' are not exactly equal, and the image is said to aberrated.



- The deviations of the converging wavefront from a spherical form are called *wave aberrations*, which can be determined by tracing rays through the imaging system. Our task is to determine the aberrated image, not of just a point object, but an extended object.

- Rays from a point object P_o passing through the entrance pupil EnP determine the flux in the Gaussian image point P_g. In reality, however, the image is not a point because of diffraction, as stated earlier.

- The complex amplitude at the exit pupil ExP can be determined by tracing rays, taking into account any nonuniform transmissivity or reflectivity of the imaging elements.

- Given the complex amplitude of the wave at ExP, our task is to determine the complex amplitude in the Gaussian image plane.

- So we first consider the problem of determining the complex amplitude in a certain plane given the complex amplitude in another. We do this by studying the Rayleigh-Sommerfeld theory of diffraction.

For other theories of diffraction, such as Kirchhoff, the reader may refer to Born and Wolf* and Goodman.**

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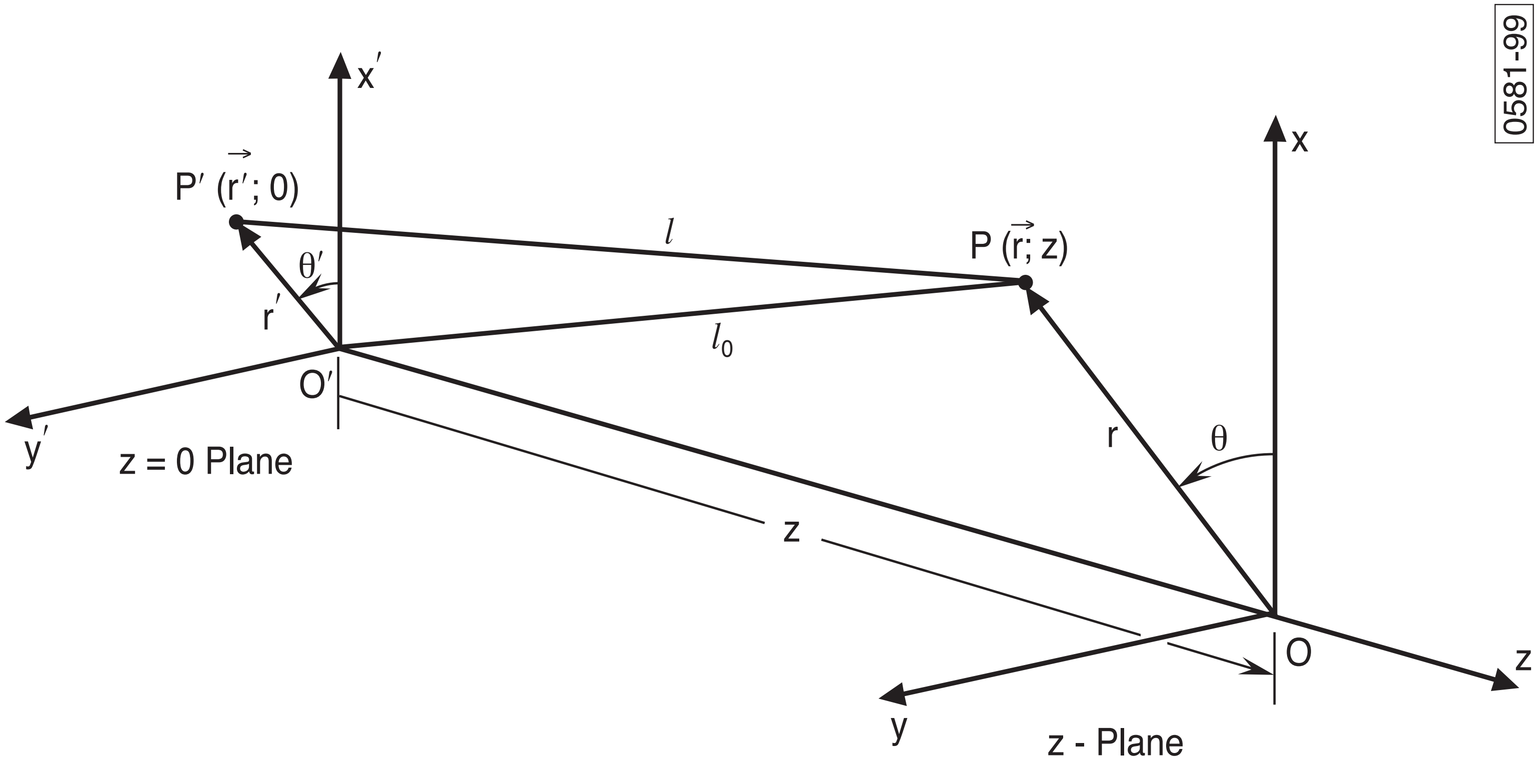
*M. Born and E. Wolf, 1999, *Principles of Optics*, Oxford, New York (1999).

**J. Goodman, *Introduction to Fourier Optics*, 2nd ed., McGraw-Hill, New York (1996).

Rayleigh-Sommerfeld Theory of Diffraction



- Given the complex amplitude $U(\vec{r}\,;0)$ in the z = 0 plane of a wave propagating in free space in the + z direction, determine the amplitude $U(\vec{r}\,;z)$ in the z-plane.

- Helmholtz scalar wave equation for free-space propagation governs how the amplitude propagates:

$$\left(\nabla^2 + k^2\right)U(\vec{r}\,;z) = 0 \quad \text{(1-1)}$$

- Decompose $U(\vec{r}\,;0)$ into plane waves:

$$U(\vec{r}\,;0) = \int A(\vec{v}\,;0)\exp(-2\pi i\vec{r}\cdot\vec{v})\,d\vec{v} \quad (U \text{ is 2D inverse FT}^{*} \text{ of } A)$$

- $A(\vec{v}\,;0)$ is the spectral component of $U(\vec{r}\,;0)$ propagating with direction cosines (α,β,γ) such that the spatial frequency $\vec{v}$ is given by

$$\vec{v} = (\alpha/\lambda,\beta/\lambda) \quad , \alpha^2+\beta^2+\gamma^2 = 1$$

- $A(\alpha/\lambda,\beta/\lambda\,;0)$ is the angular spectrum of $U(\vec{r}\,;0)$

$$A(\vec{v}\,;0) = \int U(\vec{r}\,;0)\exp(2\pi i\vec{r}\cdot\vec{v})\,d\vec{r} \quad (A \text{ is 2D FT of } U)$$

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*Read the definitions of Fourier transform and convolution given in the Appendix of Chapter 1.

Similarly,

$$U(\vec{r};z) = \int A(\vec{v};z)\exp(-2\pi i\vec{r}\cdot\vec{v})\,d\vec{v} \qquad \text{(1-7)}$$

$$A(\vec{v};z) = \int U(\vec{r};z)\exp(2\pi i\vec{r}\cdot\vec{v})\,d\vec{r}$$

- Since the wave equation is linear, each component $A(\vec{v};z)\exp(-2\pi i\vec{r}\cdot\vec{v})$ of $U(\vec{r};z)$ satisfies it:

$$\therefore \quad \left(\nabla^2 + k^2\right)A(\vec{v};z)\exp(-2\pi i\vec{r}\cdot\vec{v}) = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) A(\vec{v};z)\exp\left[-\frac{2\pi i}{\lambda}(x\alpha + y\beta)\right] = 0$$

$$\left[\frac{\partial^2 A}{\partial z^2} + A(-2\pi i)^2\frac{1}{\lambda^2}\left(\alpha^2 + \beta^2\right) + k^2\right]\exp(-2\pi i\vec{r}\cdot\vec{v}) = 0$$

$$\frac{\partial^2 A}{\partial z^2} + \gamma^2 k^2 A = 0$$

$$\therefore \quad A(\vec{v};z) = A(\vec{v};0)\exp(ik\gamma z) \qquad \text{(1-10)}$$

- *Phase of A changes, but its magnitude stays the same, as it propagates.*

- Let $h(\vec{r};z)$ be the inverse Fourier transform of $\exp(ik\gamma z)$:

$$h(\vec{r};z) = \int \exp(ik\gamma z)\exp(-2\pi i\vec{r}\cdot\vec{v})d\vec{v}$$

$$\exp(ik\gamma z) = \int h(\vec{r};z)\exp(2\pi i\vec{r}\cdot\vec{v})d\vec{r} \qquad \text{(1-13)}$$

Substitute (1-10) and (1-13) into (1-7):

$$\begin{aligned} U(\vec{r};z) &= \int h(\vec{r}\,';z)d\vec{r}\,'\int A(\vec{v};0)\exp\left[-2\pi i(\vec{r}-\vec{r}\,')\cdot\vec{v}\right]d\vec{v} \\ &= \int h(\vec{r}\,';z)U(\vec{r}-\vec{r}\,';0)d\vec{r}\,' \\ &= \int h(\vec{r}-\vec{r}\,';z)U(\vec{r}\,';0)d\vec{r}\,' \qquad \text{(1-15)} \end{aligned}$$

- Thus, $U(\vec{r};z)$ is the *convolution* of $h(\vec{r};z)$ and $U(\vec{r};0)$

Consider a point source of unit amplitude placed at a point $\vec{r}_o$ in the $z=0$ plane:

$$U(\vec{r}\,';0) = \delta(\vec{r}\,'-\vec{r}_o)$$

$$\therefore \quad U(\vec{r};z) = h(\vec{r}-\vec{r}_o;z)$$

- Thus, $h(\vec{r};z)$ represents the complex amplitude at a point $\vec{r}$ in the z-plane due to a point source of unit amplitude located at $\vec{r}=0$ in the $z=0$ plane. It is called the *complex amplitude point-spread function* (or impulse response) of free space.

It can be shown that the inverse FT of $\exp(ik\gamma z)$ is given by*

$$h(\vec{r};z) = \int \exp(ik\gamma z)\exp(-2\pi i\vec{r}\cdot\vec{v})d\vec{v} = \frac{1}{\lambda}\left(\frac{1}{kl_0}-i\right)\frac{z}{l_0}\frac{\exp(ikl_0)}{l_0} \qquad (1\text{-}18)$$

Here, $l_0 = \left(z^2+r^2\right)^{1/2}$ is the distance $O'P$ between the origin O' in the $z=0$ plane and the observation point $P(\vec{r};z)$. We can see that $h(\vec{r};0)$ is a point source of unit amplitude, as expected:

$$h(\vec{r};0) = \int \exp(-2\pi i\vec{r}\cdot\vec{v})d\vec{v} = \delta(\vec{r})$$

- Eq. (1-18) represents *Huygens' spherical wavelet* diverging from the point source.

*E. Lalor, “Conditions for the validity of angular spectrum of plane waves,” *J. Opt. Soc. Am.* **58**, 1235–1237 (1968).

$$U(\vec{r};z) = \int h(\vec{r}-\vec{r}\,';z)U(\vec{r}\,';0)d\vec{r}\,' \quad \text{(1-15)}$$

- Eq. (1-15) is a mathematical description of *Huygens-Fresnel Principle*, namely, that the amplitude $U(\vec{r};z)$ in the z-plane is a linear superposition of Huygens' secondary wavelets $h(\vec{r}-\vec{r}\,';z)$ weighted by the amplitudes $U(\vec{r}\,';0)$ of the wave where they originate (Theorem 1).

- $U(\vec{r};z)$ is *shift invariant* or *isoplanatic* in that a spherical wavelet at a point $\vec{r}$ due to a source point at $\vec{r}\,'$ depends on $\vec{r}-\vec{r}\,'$, i.e., the form of the wavelet is independent of the location of its origin in the $z=0$ plane, except for a shift in the center of the distribution.

Substituting Eq. (1-18) into Eq. (1-15), we obtain the *Rayleigh-Sommerfeld formula* describing propagation of a wave from one plane to another:

$$U(\vec{r};z) = \frac{1}{\lambda}\int U(\vec{r}\,';0)\left(\frac{1}{kl}-i\right)\frac{z}{l}\frac{\exp(ikl)}{l}d\vec{r}\,' \quad \text{(1-20)}$$

$$l = \left[z^2+\left|\vec{r}-\vec{r}\,'\right|^2\right]^{1/2} = z+\frac{1}{2z}\left(r^2+r'^2-2\vec{r}\cdot\vec{r}\,'\right)-\frac{1}{8z^3}\left|\vec{r}-\vec{r}\,'\right|^4+\ldots$$

is the distance $P'P$ between a source point $P'(\vec{r}\,';0)$ and the observation point $P(\vec{r};z)$.

Rayleigh-Sommerfeld region:

Very small values of z (e.g., very close to ExP)

Fresnel approximation:

Moderately large z, $kl >> 1$, $l \simeq z$, obliquity factor $z/l \simeq 1$ (represents the cosine of the angle $P'P$ makes with the z axis). In the exponent,

$$l \; = \; z + \frac{1}{2z}\left(r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'\right) \qquad \text{for } z^3 > \frac{k\left|\vec{r} - \vec{r}'\right|^4_{\max}}{8} \quad \text{(Fresnel Condition)}$$

$$\therefore \; U(\vec{r}\,;z) \; = \; \frac{-i}{\lambda z}\exp\left[ik\left(z + \frac{r^2}{2z}\right)\right]\int \exp\left(\frac{ikr'^2}{2z}\right)U(\vec{r}\,';0)\exp\left(-\frac{2\pi i}{\lambda z}\vec{r}\cdot\vec{r}'\right)d\vec{r}\,' \quad \text{(1-22a)}$$

kz is the phase accumulated in propagating a distance z

$(k/2z)r^2$ is a quadratic phase factor outside the integral

$(k/2z)r'^2$ is a quadratic phase factor (similar to defocus aberration) inside the integral

Fraunhofer approximation:

Very large values of z, we neglect the quadratic phase factor $(k/2z)r'^2$:

$$l = z + \frac{1}{2z}\left(r^2 - 2\vec{r}\cdot\vec{r}'\right) \quad \text{for } z > \frac{kr'^2_{\max}}{2} \quad \text{(Fraunhofer Condition)}$$

$$\therefore \quad U(\vec{r};z) = \frac{-i}{\lambda z}\exp\left[ik\left(z+\frac{r^2}{2z}\right)\right]\int U(\vec{r}';0)\exp\left(-\frac{2\pi i}{\lambda z}\vec{r}\cdot\vec{r}'\right)d\vec{r}' \qquad \text{(1-23a)}$$

- Thus, the *diffracted amplitude is a Fourier transform of the initial amplitude with a quadratic phase factor in the Fresnel approximation and without the quadratic phase factor in the Fraunhofer approximation* (Theorem 2).

- Note that Rayleigh-Sommerfeld diffraction encompasses Fresnel and Fraunhofer diffraction, and Fresnel diffraction encompasses Fraunhofer diffraction.

Numerical example:

Pupil of diameter D = 1 cm, $r'_{\max} = 0.5\,\mathrm{cm}$, $\lambda = 0.5\,\mu\mathrm{m}$

Fresnel:

$$z^3 > \frac{k\,r'^{4}_{\max}}{8} = 982 \Rightarrow z > 10\,\mathrm{cm}$$

Fraunhofer:

$$z > \frac{kr'^{2}_{\max}}{2} = 157\,\mathrm{m} \quad \text{(Far-field distance)}$$

- In imaging problems, the far-field distance is reduced to the distance of the image plane from the exit pupil.

- Hence the diffracted amplitude in the image plane will be given by a Fourier transform of the amplitude in the pupil plane.

- However, if the image is observed in a defocused image plane, then it will be the Fourier transform of the "defocused" amplitude.

- Thus, we will deal with both Fresnel and Fraunhofer diffraction.

PSF and OTF of Free Space

Point-spread function (PSF): $h(\vec{r}\,;z)$

$$h(\vec{r}\,;z) = \frac{1}{\lambda}\left(\frac{1}{kl_0} - i\right)\frac{z}{l_0}\frac{\exp\left(ikl_0\right)}{l_0}$$ (Rayleigh-Sommerfeld, Huygens' spherical wavelet)

$$= -\frac{i}{\lambda z}\exp\left[ikz\left(1 + \frac{r^2}{2z^2}\right)\right]$$ (Fresnel; spherical→ parabolic wavelet)

$$h(\vec{r} - \vec{r}\,';z) = -\frac{i}{\lambda z}\exp\left\{ikz\left[1 + \frac{(\vec{r} - \vec{r}\,')^2}{2z^2}\right]\right\}$$

For a point source at $\vec{r}\,'$,

$$\therefore\ h(\vec{r}\,;\vec{r}\,';z) = -\frac{i}{\lambda z}\exp\left[ikz\left(1 + \frac{r^2 - 2\vec{r}\cdot\vec{r}\,'}{2z^2}\right)\right]$$ (Fraunhofer; spherical→planar)

Optical transfer function (OTF): $H(\alpha, \beta; z)$

OTF $\exp(ik\gamma z)$ is FT of PSF $h(\vec{r}; z)$, where $\gamma = \left(1 - \alpha^2 - \beta^2\right)^{1/2}$

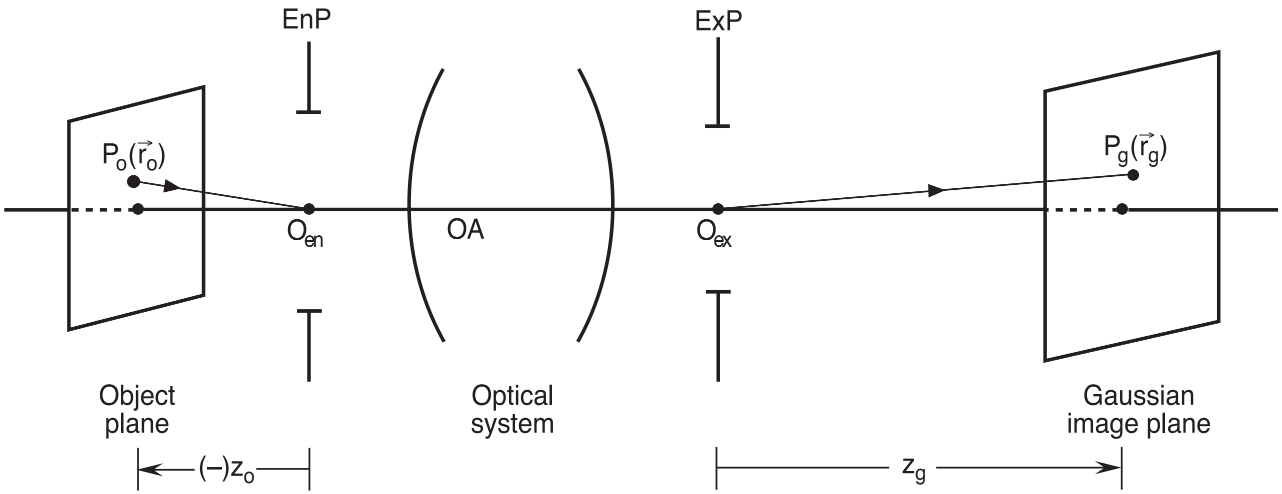
$$A(\vec{v}; z) = A(\vec{v}; 0)\exp(ik\gamma z) \qquad (1\text{-}10)$$

- *output spectrum* $A(\vec{v}; z)$ *is equal to the product of the input spectrum* $A(\vec{v}; 0)$ *and the OTF* $\exp(ik\gamma z)$

$$H(\alpha, \beta; z) = \exp\left[ikz\left(1 - \alpha^2 - \beta^2\right)^{1/2}\right] \quad \text{(Rayleigh} - \text{Sommerfeld)} \qquad (1\text{-}27)$$

$$= \exp\left\{ikz\left[1 - \frac{1}{2}\left(\alpha^2 + \beta^2\right)\right]\right\} \quad \text{(Fresnel and Fraunhofer)} \qquad (1\text{-}28)$$

Gaussian Image



- Gaussian image of an object is its aberration-free image based on geometrical optics.

Object point P_o location: $\vec{r}_o$

Transverse magnification of the image: M

Image point P_g location: $\vec{r}_g = M\vec{r}_o$

Object element area: $\Delta\vec{r}_o$

Image element area: $\Delta\vec{r}_g = M^2\Delta\vec{r}_o$

Object radiance: $B(\vec{r}_o)$ in $\mathrm{W/m^2\,sr}$

Object element power entering EnP of area S_{en} at a distance z_o:

$$P_{en} = \left(S_{en}/z_o^2\right)B(\vec{r}_o)\Delta\vec{r}_o$$

Image power exiting from ExP and contained in the image element of irradiance $I_g(\vec{r}_g)$ for a power transmission factor η from EnP to ExP:

$$P_{ex} = \eta P_{en} = I_g(\vec{r}_g)\Delta\vec{r}_g$$

$$I_g(\vec{r}_g)\Delta\vec{r}_g = \eta\left(S_{en}/z_o^2\right)B(\vec{r}_o)\Delta\vec{r}_o$$

$$= \eta\left(S_{en}/z_o^2\right)B\left(\frac{\vec{r}_g}{M}\right)\frac{\Delta\vec{r}_g}{M^2}$$

$$\therefore \quad I_g(\vec{r}_g) = \eta \frac{S_{en}}{z_o^2 M^2} B\left(\frac{\vec{r}_g}{M}\right) \quad (1\text{-}35)$$

- The *Gaussian image is an exact replica of the object, except for magnification* (Theorem 3).

- In reality, the illumination generally decreases as $\cos^4\theta$, where θ is the field angle.

- There is additional decrease due to vignetting of rays (See Section 2.2.4 of Part I*).

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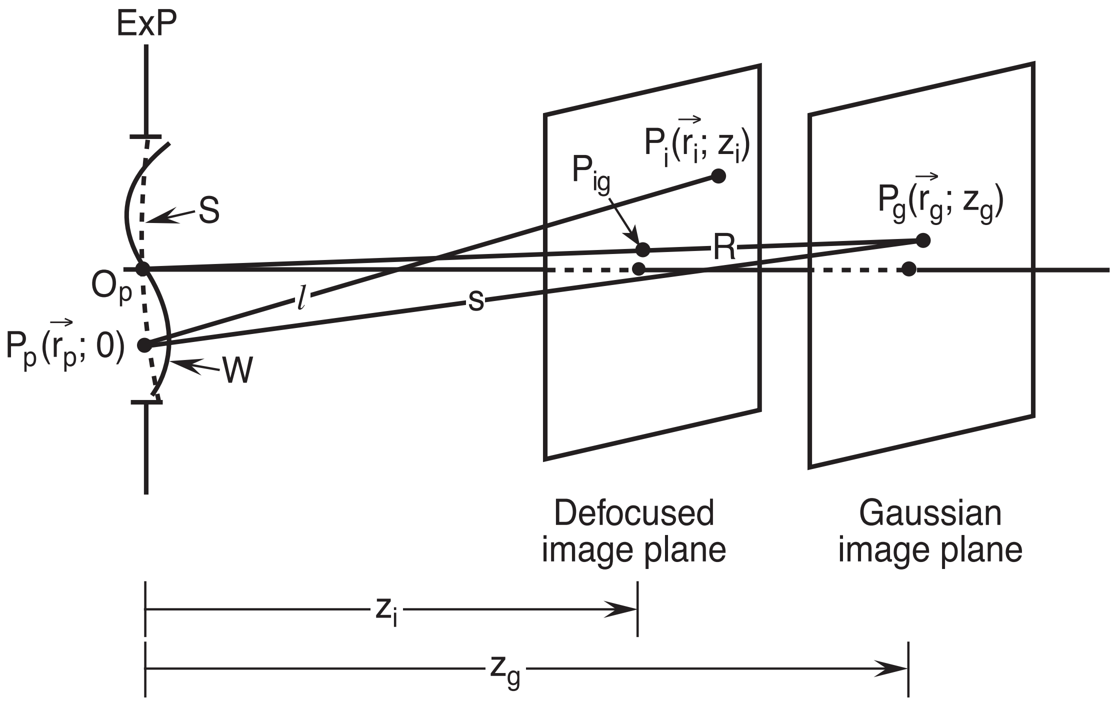
*V. N. Mahajan, “Optical Imaging and Aberrations, Part I: Ray Geometrical Optics” (1998, SPIE Press).

Diffraction Image

- In Gaussian optics, a spherical wave diverging from the point object is converted into a spherical wave converging to the Gaussian image point.
- In practice, an imaging system converts the spherical wave diverging from the point object into an *approximate* spherical wave converging to the Gaussian image point.



- A sphere passing through the center of ExP with center of curvature at the Gaussian image point is referred to as the *Gaussian reference sphere S.*

- *Wave aberration* of a ray represents its optical path length relative to that of the chief ray from the object point to the reference sphere. Let $W\left(\vec{r}_p;\vec{r}_o\right)$ be the wave aberration of a ray at a point $\vec{r}_p$ in ExP plane from a point object located at $\vec{r}_o$.

Complex amplitude at a point $P_p\left(\vec{r}_p\right)$ in the plane of ExP (aberrated and apodized spherical wave centered at $P_g\left(\vec{r}_g\right)$) due to an object element $\Delta\vec{r}_o$:

$$\Delta U_{ex}\left(\vec{r}_p;\vec{r}_o\right) = P\left(\vec{r}_p;\vec{r}_o\right)\exp(-iks) \quad \text{(1-36)}$$

$$\begin{aligned} P\left(\vec{r}_p;\vec{r}_o\right) &= A\left(\vec{r}_p;\vec{r}_o\right)\exp\left[ikW\left(\vec{r}_p;\vec{r}_o\right)\right] \quad , \text{ inside the exit pupil} \\ &= 0 \quad , \text{ outside the exit pupil} \end{aligned} \quad \text{(1-38)}$$

$A\left(\vec{r}_p;\vec{r}_o\right)$ is an *apodization function* representing nonuniform transmission of the imaging system; it also contains the inverse-square law dependence on distance s.

Substitute (1-36) into Rayleigh-Sommerfeld formula (1-20), neglect $1/kl$ and the obliquity factor z/l, replace l by z, except in the exponent:

$$\Delta U_i\left(\vec{r}_i;\vec{r}_o;z_i\right) = -\frac{i}{\lambda z_i}\int P\left(\vec{r}_p;\vec{r}_o\right)\ \exp\left[ik\left(l-s\right)\right]d\vec{r}_p$$

Distance between pupil point $P_p\left(\vec{r}_p\right)$ and the observation image point $P_i\left(\vec{r}_i;z_i\right)$:

$$l = P_pP_i = \left[z_i^2+\left|\vec{r}_p-\vec{r}_i\right|^2\right]^{1/2}$$

$$= z_i+\frac{1}{2z_i}\left(r_p^2+r_i^2-2\vec{r}_p\cdot\vec{r}_i\right)+....$$

Distance between pupil point $P_p\left(\vec{r}_p\right)$ and the Gaussian image point $P_g\left(\vec{r}_g;z_g\right)$:

$$s = P_pP_g$$

$$= \left[z_g^2+\left|\vec{r}_p-\vec{r}_g\right|^2\right]^{1/2}$$

$$= z_g+\frac{1}{2z_g}\left(r_p^2+r_g^2-2\vec{r}_p\cdot\vec{r}_g\right)+....$$

$$l - s = (z_i - z_g) + \frac{1}{2}\left(\frac{r_i^2}{z_i} - \frac{r_g^2}{z_g}\right) - \frac{1}{z_i}\vec{r}_p \cdot \left(\vec{r}_i - \frac{z_i}{z_g}\vec{r}_g\right) + \frac{1}{2}\left(\frac{1}{z_i} - \frac{1}{z_g}\right) r_p^2 + \ldots$$

$$\Delta U_i(\vec{r}_i;\vec{r}_o;z_i) = -\frac{i}{\lambda z_i}\exp\left\{ik\left[(z_i - z_g) + \frac{1}{2}\left(\frac{r_i^2}{z_i} - \frac{r_g^2}{z_g}\right)\right]\right\}$$

$$\times \int P(\vec{r}_p;\vec{r}_o;z_i)\exp\left[-\frac{2\pi i}{\lambda z_i}\vec{r}_p \cdot \left(\vec{r}_i - \frac{z_i}{z_g}\vec{r}_g\right)\right] d\vec{r}_p$$

$$P(\vec{r}_p;\vec{r}_o;z_i) = P(\vec{r}_p;\vec{r}_o)\exp\left[\frac{ik}{2}\left(\frac{1}{z_i} - \frac{1}{z_g}\right) r_p^2\right]$$ (defocused pupil function)

Additional phase factor when imaging an extended object due to relative optical path length of a chief ray from a point object relative to that for an axial point object (see Figure on p. 1-18):

$$W(\vec{r}_o;\vec{r}_g) = (P_oO_{en} + z_o) + (O_{ex}P_g - z_g) \simeq -\frac{r_o^2}{2z_o} + \frac{r_g^2}{2z_g}$$

$$\Delta U_i(\vec{r}_i;\vec{r}_o;z_i) = -\frac{i}{\lambda z_i}\exp\left\{ik\left[(z_i - z_g) + \frac{1}{2}\left(\frac{r_i^2}{z_i} - \frac{r_o^2}{z_o}\right)\right]\right\}$$

$$\times \int P(\vec{r}_p;\vec{r}_o;z_i)\exp\left[-\frac{2\pi i}{\lambda z_i}\vec{r}_p\cdot\left(\vec{r}_i - \frac{z_i}{z_g}\vec{r}_g\right)\right]d\vec{r}_p \qquad (1\text{-}49)$$

- Fresnel diffraction integral of the pupil function (which is equivalent to a defocused Fraunhofer diffraction integral)

Irradiance distribution of the image element:

$$\Delta I_i(\vec{r}_i;\vec{r}_o;z_i) = \left|\Delta U_i(\vec{r}_i;\vec{r}_o;z_i)\right|^2 = P_{ex}(\vec{r}_o)PSF(\vec{r}_i;\vec{r}_o;z_i)$$

where

$$PSF(\vec{r}_i;\vec{r}_o;z_i) = \frac{1}{P_{ex}(\vec{r}_o)\lambda^2 z_i^2}\left|\int P(\vec{r}_p;\vec{r}_o;z_i)\exp\left[-\frac{2\pi i}{\lambda z_i}\vec{r}_p\cdot\left(\vec{r}_i - \frac{z_i}{z_g}M\vec{r}_o\right)\right]d\vec{r}_p\right|^2 \qquad (1\text{-}51)$$

- *Incoherent PSF* is *proportional to the modulus square of the inverse Fourier transform of the defocused pupil function of the system* (Theorem 4).

To obtain the image of an extended *incoherent object*, we add the irradiance images of its elements, while neglecting any variation of S_{en} and the solid angle it subtends at the various object elements (equivalent to assuming a small object with a small field angle):

$$I_i(\vec{r}_i;z_i) = \int \Delta I(\vec{r}_i;\vec{r}_o;z_i) = \eta\left(S_{en}/z_o^2\right) \int_{object} B(\vec{r}_o)\, PSF(\vec{r}_i;\vec{r}_o;z_i)\, d\vec{r}_o \quad (1\text{-}52)$$

Isoplanatic object: small enough object so that the pupil function and the transverse magnification are independent of the position $\vec{r}_o$ of an object element of the extended object. Thus, $z_g \simeq R$, and $z_i \simeq z_g$ if the depth of focus is small:

$$P(\vec{r}_p;\vec{r}_o;z_i) = P(\vec{r}_p;z_i) \quad \text{(Isoplanatic pupil function)}$$

$$PSF(\vec{r}_i;\vec{r}_o;z_i) = \frac{1}{P_{ex}(\vec{r}_o)\lambda^2 R^2}\left|\int P(\vec{r}_p;z_i)\exp\left[-\frac{2\pi i}{\lambda R}\vec{r}_p\cdot(\vec{r}_i - M\vec{r}_o)\right]d\vec{r}_p\right|^2$$

$$\equiv PSF(\vec{r}_i - M\vec{r}_o;z_i) \quad \text{(Isoplanatic PSF)}$$

For an isoplanatic PSF, the image of an extended object may be written

$$I_i\left(\vec{r}_i; z_i\right) = \eta\left(S_{en}/z_o^2\right) \int_{object} B\left(\vec{r}_o\right) PSF\left(\vec{r}_i - M\vec{r}_o; z_i\right) d\vec{r}_o \qquad \text{(1-56a)}$$

$$= \eta\left(S_{en}/z_o^2 M^2\right) \int B\left(\vec{r}_g/M\right) PSF\left(\vec{r}_i - \vec{r}_g; z_i\right) d\vec{r}_g$$

$$= \int I_g\left(\vec{r}_g\right) PSF\left(\vec{r}_i - \vec{r}_g; z_i\right) d\vec{r}_g \qquad \text{(1-56c)}$$

- The *diffraction image of an isoplanatic incoherent object is equal to the convolution of its Gaussian image* (which is a scaled replica of the object) *and the PSF of the imaging system* (Theorem 5)

Physical Significance of PSF

Consider a point object of intensity B_j located at $\vec{r}_j$ with its Gaussian image located at $\vec{r}_g = M\vec{r}_j$:

$$B\left(\vec{r}_o\right) = B_j\,\delta\left(\vec{r}_o - \vec{r}_j\right) \quad \text{(1-58)}$$

$$P_{ex} = \eta\left(S_{en}/z_o^2\right)B_j \quad \text{(1-59b)}$$

Substitute (1-58) into (1-52):

$$I_i\left(\vec{r}_i;\vec{r}_j;z_i\right) = P_{ex}PSF\left(\vec{r}_i;\vec{r}_j;z_i\right)$$

$$= \frac{1}{\lambda^2 z_i^2}\left|\int P\left(\vec{r}_p;\vec{r}_j;z_i\right)\exp\left[-\frac{2\pi i}{\lambda z_i}\vec{r}_p\cdot\left(\vec{r}_i - \frac{z_i}{z_g}M\vec{r}_j\right)\right]d\vec{r}_p\right|^2 \quad \text{(1-60)}$$

For an isoplanatic object, substitute (1-58) into (1-56a):

$$I_i\left(\vec{r}_i;\vec{r}_j;z_i\right) = P_{ex}PSF\left(\vec{r}_i - M\vec{r}_j;z_i\right)$$

$$= \frac{1}{\lambda^2 R^2}\left|\int P\left(\vec{r}_p;z_i\right)\exp\left[-\frac{2\pi i}{\lambda R}\vec{r}_p\cdot\left(\vec{r}_i - M\vec{r}_j\right)\right]d\vec{r}_p\right|^2 \quad \text{(1-62)}$$

- We obtain *shift invariant imaging, since* $P\left(\vec{r}_p;z_i\right)$ *is independent of the point object location* $\vec{r}_j$*.* As the point object shifts from a position $\vec{r}_j$ to a position $\vec{r}_j\,'$, the image distribution shifts as a whole from being centered at $M\vec{r}_j$ to $M\vec{r}_j\,'$ without any change.

From Eq. (1-60):

$$\int PSF\left(\vec{r}_i;\vec{r}_j;z_i\right)d\vec{r}_i = \frac{1}{\lambda^2 z_i^2}\int d\vec{r}_i\left|\int P\left(\vec{r}_p;\vec{r}_j;z_i\right)\exp\left[-\frac{2\pi i}{\lambda z_i}\vec{r}_p\cdot\left(\vec{r}_i - \frac{z_i}{z_g}M\vec{r}_j\right)\right]d\vec{r}_p\right|^2$$

$$= \frac{1}{P_{ex}\lambda^2 z_i^2}\int d\vec{r}_i\int P\left(\vec{r}_p;\vec{r}_j;z_i\right)\exp\left[-\frac{2\pi i}{\lambda z_i}\vec{r}_p\cdot\left(\vec{r}_i - \frac{z_i}{z_g}M\vec{r}_j\right)\right]d\vec{r}_p$$

$$\times\int P^*\left(\vec{r}_p\,';\vec{r}_j;z_i\right)\exp\left[\frac{2\pi i}{\lambda z_i}\vec{r}_p\,'\cdot\left(\vec{r}_i - \frac{z_i}{z_g}M\vec{r}_j\right)\right]d\vec{r}_p\,'$$

$$= \frac{1}{P_{ex}\lambda^2 {z_i}^2} \int d\vec{r}_p \, P\left(\vec{r}_p ; \vec{r}_j ; z_i\right) \int d\vec{r}_p{}' \, P^*\left(\vec{r}_p{}' \; \vec{r}_j ; z_i\right)$$

$$\times \int d\vec{r}_i \, \exp\left[\frac{2\pi i}{\lambda z_i}\left(\vec{r}_p{}' - \vec{r}_p\right) \cdot \left(\vec{r}_i - \frac{z_i}{z_g} M \vec{r}_j\right)\right]$$

$$= \frac{1}{P_{ex}} \int d\vec{r}_p \, P\left(\vec{r}_p ; \vec{r}_j ; z_i\right) \int d\vec{r}_p{}' \, P^*\left(\vec{r}_p{}' \; \vec{r}_j ; z_i\right) \delta\left(\vec{r}_p{}' - \vec{r}_p\right)$$

$$= \frac{1}{P_{ex}} \int d\vec{r}_p \left| P\left(\vec{r}_p ; \vec{r}_j ; z_i\right)^2 \right|$$

$$= 1$$

Thus, *PSF represents the irradiance distribution of the image of a point object for unity total power, and its integral over space is unity. Its dimensions are* m^{-2}.

Strehl Ratio

- Central value of the irradiance image of a point object is affected by the amplitude and phase variations across the pupil. For a given total power, the central value is maximum when the amplitude and phase are uniform.

- *Strehl ratio* is the ratio of central irradiances with and without phase/amplitude variations. It describes the effect of apodization (amplitude variations) and aberrations (phase variations).

PSF: $$I(\vec{r}_i; z_i) = \frac{1}{\lambda^2 z_i^2} \left| \int P(\vec{r}_p; z_i) \exp\left(- \frac{2\pi i}{\lambda z_i} \vec{r}_p \cdot \vec{r}_i \right) d\vec{r}_p \right|^2$$

Apodized and aberrated pupil:

$$I_{aa}(0; z_i) = \frac{1}{\lambda^2 z_i^2} \left| \int P(\vec{r}_p; z_i) d\vec{r}_p \right|^2$$

Unapodized and unaberrated pupil:

$$A(\vec{r}_p) = A_0 \quad , \quad W(\vec{r}_p) = 0 \quad , \quad z_i \rightarrow z_g \quad , \quad P(\vec{r}_p; z_i) = A_0$$

$$I_{uu}(0;z_g) = \frac{1}{\lambda^2 z_g^2}\left(\int A_0\, d\vec{r}_p\right)^2 = \frac{P_{ex}S_{ex}}{\lambda^2 z_g^2} = \frac{S_{ex}}{\lambda^2 z_g^2}\int \left|P\left(\vec{r}_p; z_i\right)\right|^2 d\vec{r}_p$$

$$\frac{I_{aa}(0;z_i)}{I_{uu}(0;z_g)} = \left(\frac{z_g}{z_i}\right)^2 S_{aa}$$

$$S_{aa} = \frac{\left|\int P\left(\vec{r}_p\right) d\vec{r}_p\right|^2}{S_{ex}\int \left|P\left(\vec{r}_p\right)\right|^2 d\vec{r}_p} \le 1$$ (using Hölder's inequality) (1-187)

- *Central irradiance for an apodized-aberrated system* (*not considering the inverse-square law effect*) *is* ≤ *the corresponding value when the system is unapodized-unaberrated* (Theorem 17a)

Apodized but unaberrated system:

$$P\left(\vec{r}_p\right) = A\left(\vec{r}_p\right)$$

$$I_{au}(0;z_g) = \left(1/\lambda^2 z_g{}^2\right)\left[\int A\left(\vec{r}_p\right) d\vec{r}_p\right]^2$$

$$S_{au} = \frac{I_{au}\left(0; z_g\right)}{I_{uu}\left(0; z_g\right)}$$

$$= \frac{\left[\int A\left(\vec{r}_p\right) d\vec{r}_p\right]^2}{S_{ex} \int A^2\left(\vec{r}_p\right) d\vec{r}_p} \leq 1 \quad \text{(1-192)}$$

- Thus, amplitude variations across the pupil reduce the central irradiance. For example, a Gaussian pupil yields a smaller central irradiance than a uniform pupil.

Also

$$\frac{S_{aa}}{S_{au}} = \frac{\left|\int P\left(\vec{r}_p\right) d\vec{r}_p\right|^2}{\left[\int A\left(\vec{r}_p\right) d\vec{r}_p\right]^2} \leq 1 \quad \text{(1-193)}$$

- Thus, amplitude variations reduce the central irradiance, and phase variations (i.e., aberrations) further reduce it.

- However, the irradiance reduced by phase variations alone does not necessarily reduce any further if amplitude variations are also introduced (Theorem 17b).

- The *peak value* of the aberrated irradiance distribution of the image of a point object does not necessarily occur at the center of the reference sphere.

- However, the peak value of its unaberrated image does occur at this point regardless of the apodization (Theorem 17c). The Huygens' spherical wavelets emanating from the spherical wavefront are equidistant from this point. Hence, they interfere constructively, producing a maximum possible value at this point.

$$f\left(\vec{r}_p\right) = A\left(\vec{r}_p\right)\exp\left(-\frac{2\pi i}{\lambda z_g}\vec{r}_p \cdot \vec{r}_i\right)$$

$$\frac{I_{au}\left(\vec{r}_i\right)}{I_{au}(0)} = \frac{\left|\int f\left(\vec{r}_p\right) d\vec{r}_p\right|^2}{\left[\int \left|f\left(\vec{r}_p\right)\right| d\vec{r}_p\right]^2} \le 1 \qquad \text{(1-194)}$$

Let $\Phi = (2\pi/\lambda)W$ be the phase aberration. Ratio of the central irradiance at a distance z_i with aberration and that at the Gaussian image point without aberration:

$$\frac{I_{aa}(0; z_i)}{I_{au}(0; z_g)} = \left(\frac{z_g}{z_i}\right)^2 S$$

$$S = \frac{\left|\int A(\vec{r}_p)\exp\left[i\Phi(\vec{r}_p)\right]d\vec{r}_p\right|^2}{\left[\int A(\vec{r}_p)d\vec{r}_p\right]^2} = \left|\langle\exp(i\Phi)\rangle\right|^2 = \left|\langle\exp\left[i(\Phi - \langle\Phi\rangle)\right]\rangle\right|^2$$

$$= \langle\cos(\Phi - \langle\Phi\rangle)\rangle^2 + \langle\sin(\Phi - \langle\Phi\rangle)\rangle^2$$

$$\geq \langle\cos(\Phi - \langle\Phi\rangle)\rangle^2 \simeq \left\langle 1 - \frac{1}{2}(\Phi - \langle\Phi\rangle)^2\right\rangle^2 \quad \text{for small aberrations}$$

Aberration variance:

$$\left\langle(\Phi - \langle\Phi\rangle)^2\right\rangle = \left\langle\Phi^2 + \langle\Phi\rangle^2 - 2\Phi\langle\Phi\rangle\right\rangle = \left\langle\Phi^2\right\rangle - \langle\Phi\rangle^2 = \sigma_\Phi^2$$

Approximate expressions for a small aberration:

$$\therefore \quad S_1 \simeq \left(1 - \sigma_\Phi^2/2\right)^2 = 1 - \sigma_\Phi^2 + \frac{1}{4}\sigma_\Phi^4 \qquad \text{(1-204)} \quad \text{(Maréchal, 1947)}$$

$$= S_2 + \frac{1}{4}\sigma_\Phi^4$$

$$S_2 \simeq 1 - \sigma_\Phi^2 \qquad \text{(1-205)} \quad \text{(Nijboer, 1943)}$$

$$S_3 \simeq \exp\left(-\sigma_\Phi^2\right) \simeq 1 - \sigma_\Phi^2 + \frac{1}{2}\sigma_\Phi^4 \qquad \text{(1-206)} \quad \text{(Mahajan, 1983)}$$

$$= S_1 + \frac{1}{4}\sigma_\Phi^4$$

- *Strehl ratio for a small aberration does not depend on its type but only on its variance across the apodized pupil* (Theorem 18).

Home Work 1

1-1. *Imaging by a Thin Lens*: Consider imaging of an object by a thin lens of focal length f lying at a (numerically negative) distance z_o from it. Using Fresnel propagation from the object plane to the lens, quadratic phase approximation of the lens resulting from focusing by it, and Fresnel propagation from the lens to an observation plane at a distance z_i from it, show that the complex amplitude in the observation plane is given by Eq. (1-49):

$$U_i\left(\vec{r}_i;\vec{r}_o;z_i\right) = -\frac{i}{\lambda z_i}\exp\left\{ik\left[\left(z_i - z_0\right)+\frac{1}{2}\left(\frac{r_i^2}{z_i}-\frac{r_o^2}{z_o}\right)\right]\right\}$$

$$\times\int P\left(\vec{r}_p;\vec{r}_o;z_i\right)\exp\left[-\frac{2\pi i}{\lambda z_i}\vec{r}_p\cdot\left(\vec{r}_i-\frac{z_i}{z_g}\vec{r}_g\right)\right]d\vec{r}_p \quad .$$