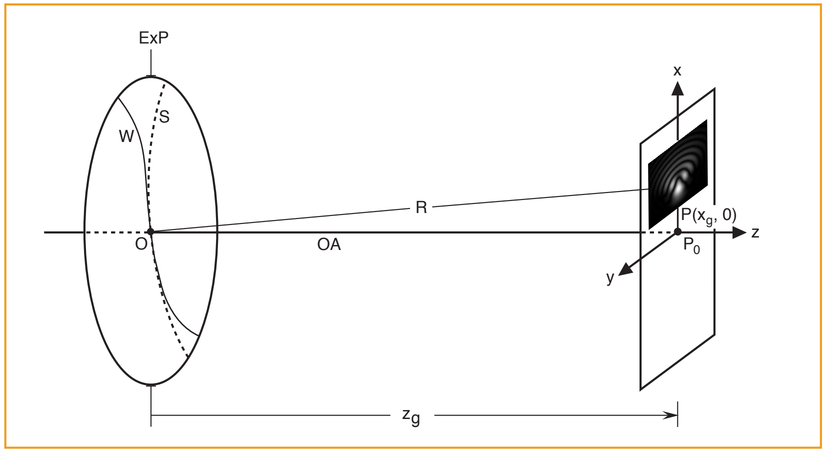
Optical Imaging and Aberrations



© Virendra N. Mahajan

Adjunct Professor
College of Optical Sciences
University of Arizona

The Aerospace Corporation
El Segundo, California 90245
(310) 336-1783

virendra.n.mahajan@aero.org

Lecture 14. Imaging With Gaussian Pupils:

Weakly-Truncated Gaussian Beams, and Line of Sight

Lecture 14 Summary

Weakly-Truncated Gaussian Beams, LOS

- Weakly-Truncated Gaussian Beam
- PSF
- Axial Irradiance
- Collimated beam
- OTF
- Strehl Ratio
- Line of Sight

Weakly-Truncated Gaussian Beam (Pupil)

Gaussian pupil is considered *weakly truncated* if the truncation parameter $\gamma = (a/\omega)^2$ is very large, i.e. if $\omega << a$.

For a weakly-truncated pupil, we may let $a \to \infty$ in the radial integrals and neglect $\exp(-2\gamma)$ compared to unity.

$$A(\rho) = A_0 \exp\left(-\gamma \rho^2\right) \quad , \quad 0 \le \rho \to \infty$$

Power incident on the pupil plane:

$$P_{inc} = \frac{S_{ex} A_0^2}{2\gamma} \quad , \quad \therefore \quad A_0^2 = 2\gamma \frac{P_{inc}}{S_{ex}}$$

Power transmitted by the pupil:

$$P_{ex} = P_{inc}\left[1 - \exp(-2\gamma)\right] \simeq P_{inc} \quad \text{(since } \gamma \text{ is large))}$$

Fractional transmitted power:

$P_{trans} = P_{ex}/P_{inc} = 1 - \exp(-2\gamma) \simeq 1$, $P_{trans} = 99.97\%$ when $\sqrt{\gamma} = a/\omega = 2$.

- Power $\exp(-2\gamma)$ lying outside the pupil is negligible when $\sqrt{\gamma} \ge 2$ or $a \ge 2\omega$.

Pupil irradiance $I(\rho) = A^2(\rho)$ (in units of P_{ex}/S_{ex}):

$$I(\rho) = \frac{2\gamma \exp\left(-2\gamma\rho^2\right)}{1 - \exp(-2\gamma)} \simeq 2\gamma \exp\left(-2\gamma\rho^2\right) \quad \text{or} \quad I_i\left(r_p\right) = \frac{2P_{ex}}{\pi\omega^2} \exp\left(-\frac{2r_p^2}{\omega^2}\right)$$

Aberration-Free PSF:

$$I(r;\gamma) = 4\left[\int_0^1 \sqrt{I(\rho)}\, J_0(\pi r\rho)\rho\, d\rho\right]^2 \quad \text{(in units of } P_{ex}S_{ex}/\lambda^2R^2\text{)}$$

Defocused PSF at a distance z (with r in units of $\lambda z/D$):

$$I(r;z;\gamma) = \left(\frac{2R}{z}\right)^2 \left[\int_0^1 \sqrt{I(\rho)}\, \exp\left(iB_d\rho^2\right) J_0(\pi r\rho)\rho\, d\rho\right]^2$$

$$\simeq 8\gamma\left(\frac{R}{z}\right)^2 \left[\int_0^\infty \exp\left(-\gamma\rho^2\right) \exp\left(iB_d\rho^2\right) J_0(\pi r\rho)\rho\, d\rho\right]^2$$

$$B_d = \frac{\pi}{\lambda}\left(\frac{1}{z} - \frac{1}{R}\right)a^2 = \pi N\left(\frac{R}{z} - 1\right) \quad , \quad N = \frac{a^2}{\lambda R} \quad \text{(Fresnel number)}$$

Letting $\beta = \pi r$ and $\alpha = \gamma - i B_d$, and utilizing

$$\int_0^\infty \left(-\alpha\rho^2\right) J_0(\beta\rho)\,\rho\,d\rho = (1/2\alpha)\exp\left(-\beta^2/4\alpha\right) \quad , \quad \mathrm{Re}\,\alpha > 0$$ yields

$$I(r;z) = \left(\frac{R}{z}\right)^2 \frac{2\gamma}{B_d^2 + \gamma^2} \exp\left[-\frac{\gamma\pi^2 r^2}{2\left(B_d^2 + \gamma^2\right)}\right]$$ or $$I_i(r_i;z) = \frac{2P_{ex}}{\pi\omega_z^2}\exp\left(-\frac{2r_i^2}{\omega_z^2}\right)$$

where $\omega_z^2 = (\lambda z/\pi\omega)^2 + \omega^2(1 - z/R)^2$

- Thus *a weakly-truncated Gaussian beam expands but remains Gaussian as it propagates.*

- Letting $z = 0$ yields the pupil irradiance distribution.

Focal-plane distribution:

$$I(r;R) = \frac{2}{\gamma}\exp\left(-\frac{\pi^2 r^2}{2\gamma}\right)$$ or $$I_i(r_i, R) = \frac{2P_{ex}}{\pi\omega_R^2}\exp\left(-\frac{2r_i^2}{\omega_R^2}\right) \quad , \quad \omega_R = \frac{\lambda R}{\pi\omega}$$

$$I(0;R) = \frac{2}{\gamma}$$ or $$I_i(0, R) = \frac{2P_{ex}}{\pi\omega_R^2}$$

PSF and encircled-power distributions for $\sqrt{\gamma} = 2$

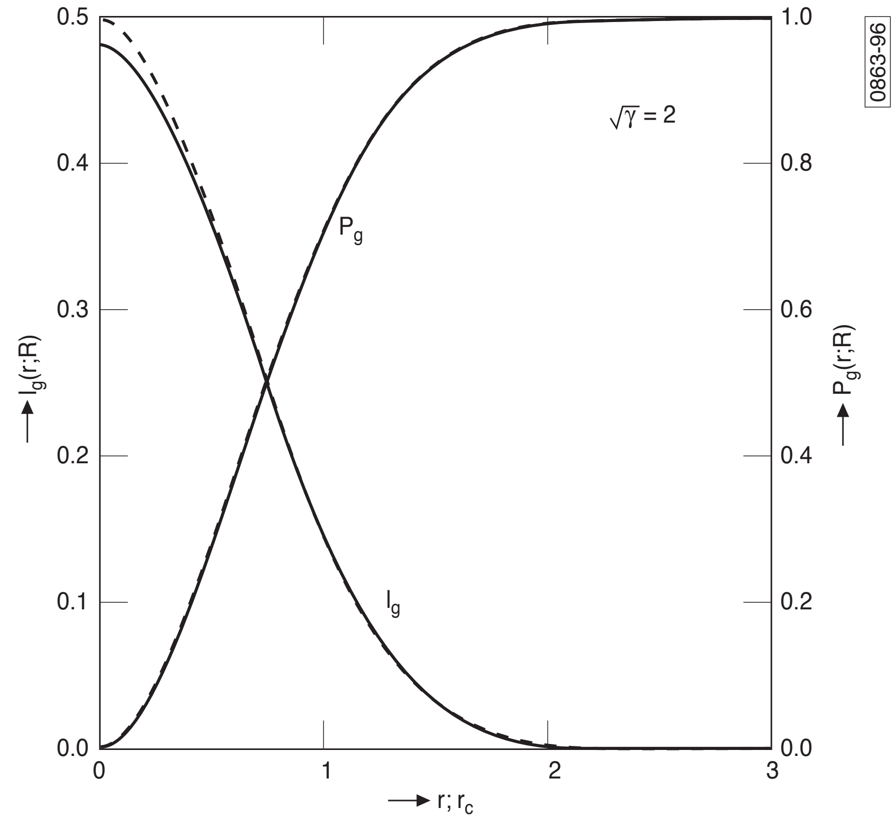


Figure 4-9. Focal-plane irradiance (in units of $P_{ex}S_{ex}/\lambda^2 R^2$) and encircled-power distributions for a Gaussian beam with $\sqrt{\gamma} = 2$. r and r_c in units of λF.

• Solid curves represent the exact results and the dashed curves represent their corresponding approximations by neglecting the beam truncation.

* Approximate results agree well with the exact results.

• Maximum difference, which occurs at the focus, is less than 4%.

• Approximate result overestimates the central irradiance.

• For larger γ, the agreement is found to be even better.

• Hence, it is reasonable to neglect beam truncation when $\sqrt{\gamma} \geq 2$.

• **However**, when the beam is aberrated, a larger value of γ, namely, $\sqrt{\gamma} \geq 3$, is required for the validity of weakly-truncated approximation. This is discussed later under Strehl ratio.

Axial Irradiance:

$$I(0;z) = \left(\frac{R}{z}\right)^2 \frac{2\gamma}{B_d^2 + \gamma^2} \qquad \text{or} \qquad I_i(0;z) = \frac{2P_{ex}}{\pi\omega_z^2}$$

Pupil irradiance at its center:

$$I(0;0) = \frac{2\gamma}{\pi^2 N^2} \qquad \text{(in units of } P_{ex}S_{ex}/\lambda^2 R^2\text{)}$$

$$I_i(0;0) = \frac{P_{ex}S_{ex}}{\lambda^2 R^2}\frac{2\gamma}{\pi^2 N^2} = \frac{2P_{ex}}{\pi\omega^2} = A_0^2 \qquad \text{(since } P_{ex} \simeq P_{inc}\text{)}$$

Peak axial irradiance:

Letting $\partial I(0;z)/\partial z = 0$ yields the location z_p of peak irradiance:

$$\frac{z_p}{R} = \frac{1}{1 + (\gamma/\pi N)^2} \qquad \text{or} \qquad \frac{z_p}{R} = \frac{1}{1 + (\omega_R/\omega)^2} < 1$$

• Thus, the *peak irradiance does not occur at the focal point but at a point closer to the pupil.*

Peak value:

$$I(0; z_p) = \frac{2}{\gamma} + \frac{2\gamma}{\pi^2 N^2} \quad \text{or} \quad I_i(0; z_p) = \frac{2 P_{ex}}{\pi \omega_{zp}^2}$$

Peak irradiance = Irradiance at pupil center + Focal-point irradiance

Beam waist:

$$\omega_{zp}^2 = \frac{\omega^2}{1 + (\omega/\omega_R)^2} = \frac{\omega_R^2}{1 + (\omega_R/\omega)^2} \Rightarrow \omega_{zp} < \omega \text{ or } \omega_R$$

• Even though the peak axial irradiance and the beam waist are not located at the focal point $(z = R)$, the *smallest beam radius and maximum central irradiance on a target at a fixed distance* z *are obtained when the beam is focused on it*.

Encircled-power:

$$P(r_c; z) = 1 - \exp\left[- \gamma\pi^2 r_c^2 \big/ 2\left(B_d^2 + \gamma^2\right)\right]$$ (r_c is in units of $\lambda z/D$)

or

$$P_i(r_c; z) = P_{ex}\left\{1 - \exp\left[\left(- 2 r_c^2 \big/ \omega_z^2\right)\right]\right\}$$

Axial irradiance of a Gaussian beam for $\sqrt{\gamma} = 2$ and $N = 1$, 10, 100

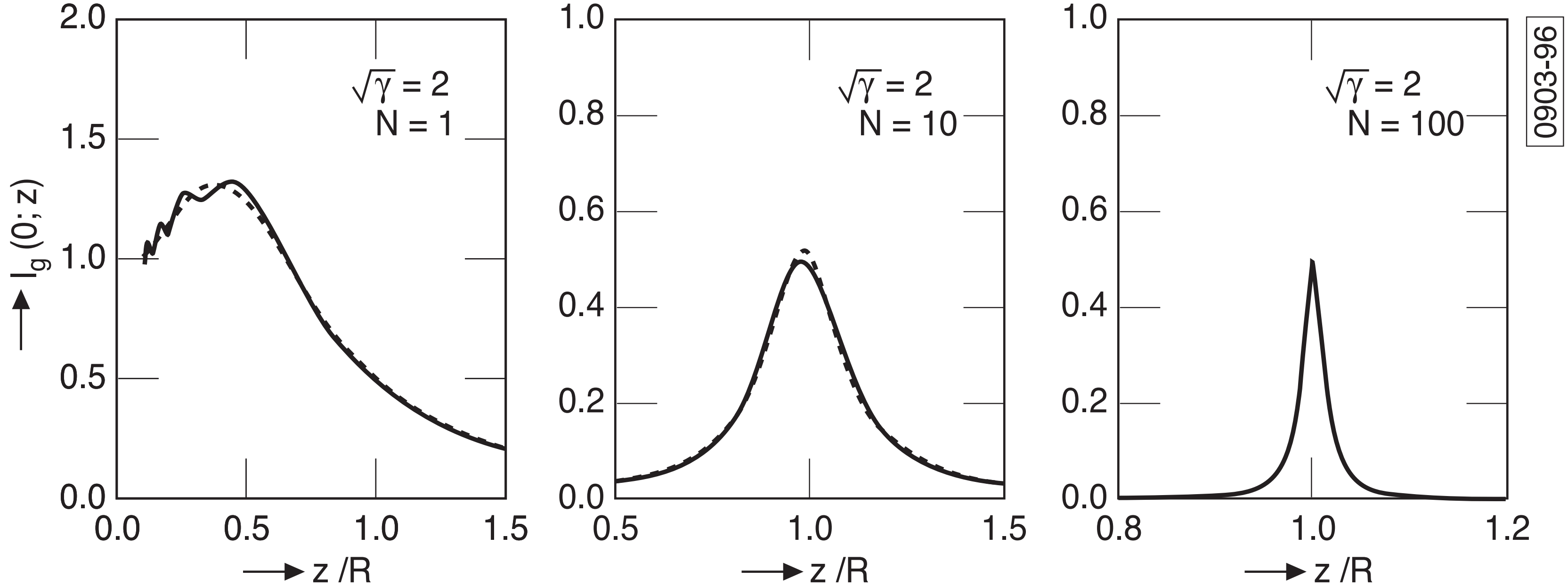
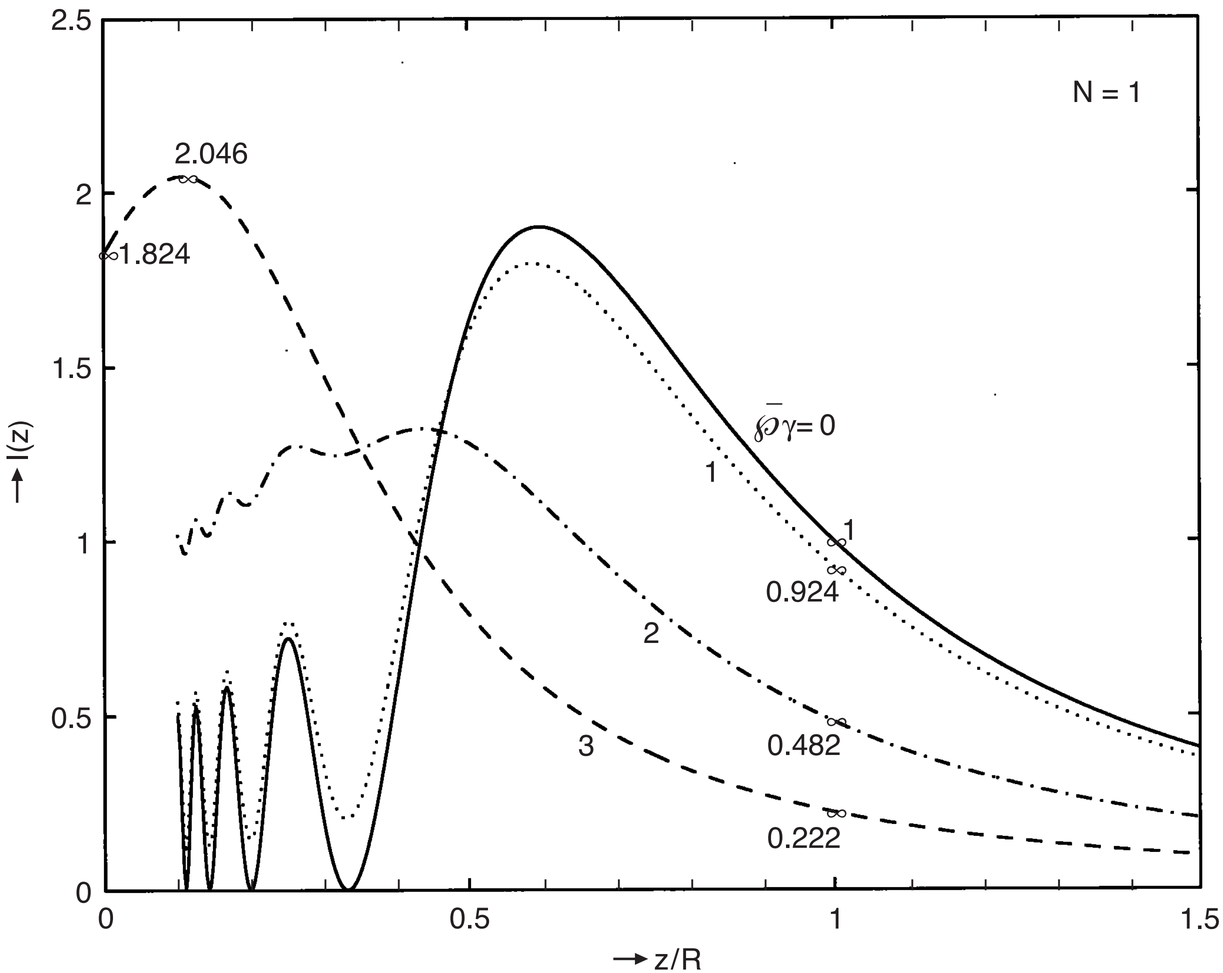


Figure 4-10. Axial irradiance (in units of $P_{ex}S_{ex}/\lambda^2 R^2$) of a Gaussian beam.

* Solid curves represent the exact results and the dashed curves represent their corresponding approximations by neglecting the beam truncation.

• Approximate results agree well with the exact results, except when $N = 1$ in that the exact results show secondary maxima and minima, but the approximate result shows only the principal maximum.



Axial irradiance (in units of $P_{ex}S_{ex}/\lambda^2 R^2$) of a Gaussian beam with various truncation ratios focused at a distance R with a Fresnel number $N = 1$.

Radius of curvature of the spherical wavefront at a distance z

• Complex amplitude of an apertured converging spherical wave of radius of curvature R in the quadratic (or Fresnel) approximation varies as $\exp\left(-i\pi r_i^2/\lambda R\right)$, where r_i is the radial distance of a point in the pupil plane from its axis.

• Thus, the radius of curvature of a wavefront is given by the inverse of the coefficient of $-i\pi r_i^2/\lambda$ in the exponent of its complex amplitude representation.

• Diffracted amplitude:

$$U(\vec{r}_i;z) = -2i\sqrt{2\gamma}\left(\frac{R}{z}\right)\exp\left[ik\left(z+\frac{1}{2}\frac{r_i^2}{z}\right)\right]\int_0^\infty \exp\left[-(\gamma-iB_d)\rho^2\right]J_0(\pi\rho\, r_i D/\lambda z)\rho\, d\rho$$

$$= -\sqrt{2\gamma}\left(\frac{R}{z}\right)\frac{i}{(\gamma-iB_d)}\exp\left[ik\left(z+\frac{1}{2}\frac{r_i^2}{z}\right)\right]\exp\left[-(\pi r_i D/\lambda z)^2/4(\gamma-iB_d)\right]$$

Phase factor varying as r_i^2: $$\left[\frac{ik}{2z} - \frac{(\pi D/\lambda z)^2}{4(\gamma - iB_d)}\right] r_i^2$$

Phase factor varying as ir_i^2: $$i\left[\frac{k}{2z} + - \frac{(\pi D/\lambda z)^2 B_d}{4\left(\gamma^2 + B_d^2\right)}\right] r_i^2$$

Radius of curvature R_z of the wavefront at a distance z:

$$\frac{z}{R_z} = \frac{S_{ex} B_d}{\lambda z\left(B_d^2 + \gamma^2\right)} - 1 = \frac{1 - z/R}{(1 - z/R)^2 + \left(\lambda z/\pi \omega^2\right)^2} - 1$$

• At the waist position z_p, $R_{z\,p} = \infty$, implying a plane wave.

• At the focal plane, $R_z = -R$; a negative sign of R_z indicates a diverging spherical wave.

• For $z > R$, $\dfrac{z}{R_z} < -1$, and the beam continues to expand as it propagates.

Collimated Beam

Letting $R \to \infty$ yields $B_d = S_{ex}/\lambda z$ and

$$I(r;z) = \frac{2\gamma}{1+(4\gamma z/\pi)^2}\exp\left[\frac{-8\gamma z^2 r^2}{1+(4\gamma z/\pi)^2}\right] \quad \text{(in units of } P_{ex}/S_{ex}\text{)}$$

r in units of $\lambda z/D$ (z is not normalized here), and z is in units of D^2/λ.

or $$I_i(r_i;z) = \frac{2P_{ex}}{\pi\omega_z^2}\exp\left(-\frac{2r_i^2}{\omega_z^2}\right) \quad , \quad \omega_z^2 = \omega^2\left[1+\left(\frac{\lambda z}{\pi\omega^2}\right)^2\right]$$

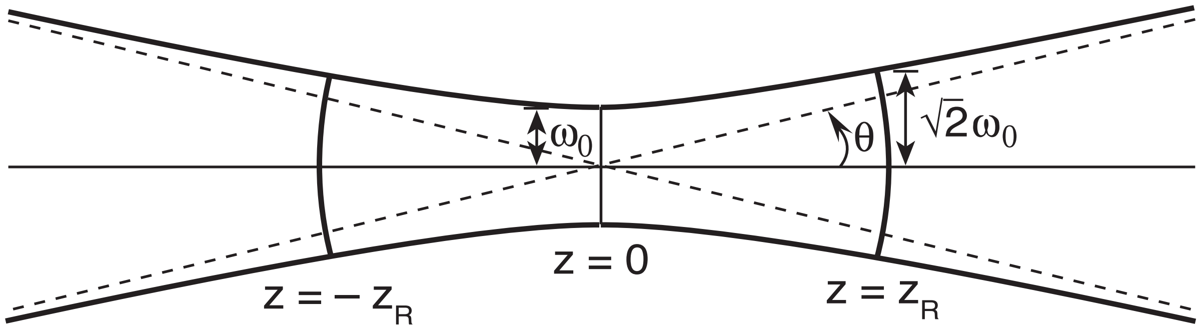
$$R_z = -z\left[1+\left(\pi\omega^2/\lambda z\right)^2\right] = -z\left[1+\left(z_R/z\right)^2\right]$$

• ω_z increases and irradiance decreases monotonically as z increases; i.e., the beam expands as it propagates.

Beam radius at a distance z may also be written

$$\omega_z^2 = \omega^2\left[1+\left(z/z_R\right)^2\right] \quad \text{where} \quad z_R = \pi\omega^2/\lambda$$

is a distance, called the Rayleigh range, from the plane of the beam waist to a plane in which the beam radius has increased by a factor of $\sqrt{2}$ and, therefore, the axial irradiance has decreased by a factor of 2.



Beam radius and divergence angle. The radius of the beam is ω_0 and its wavefront is planar at $z = 0$.

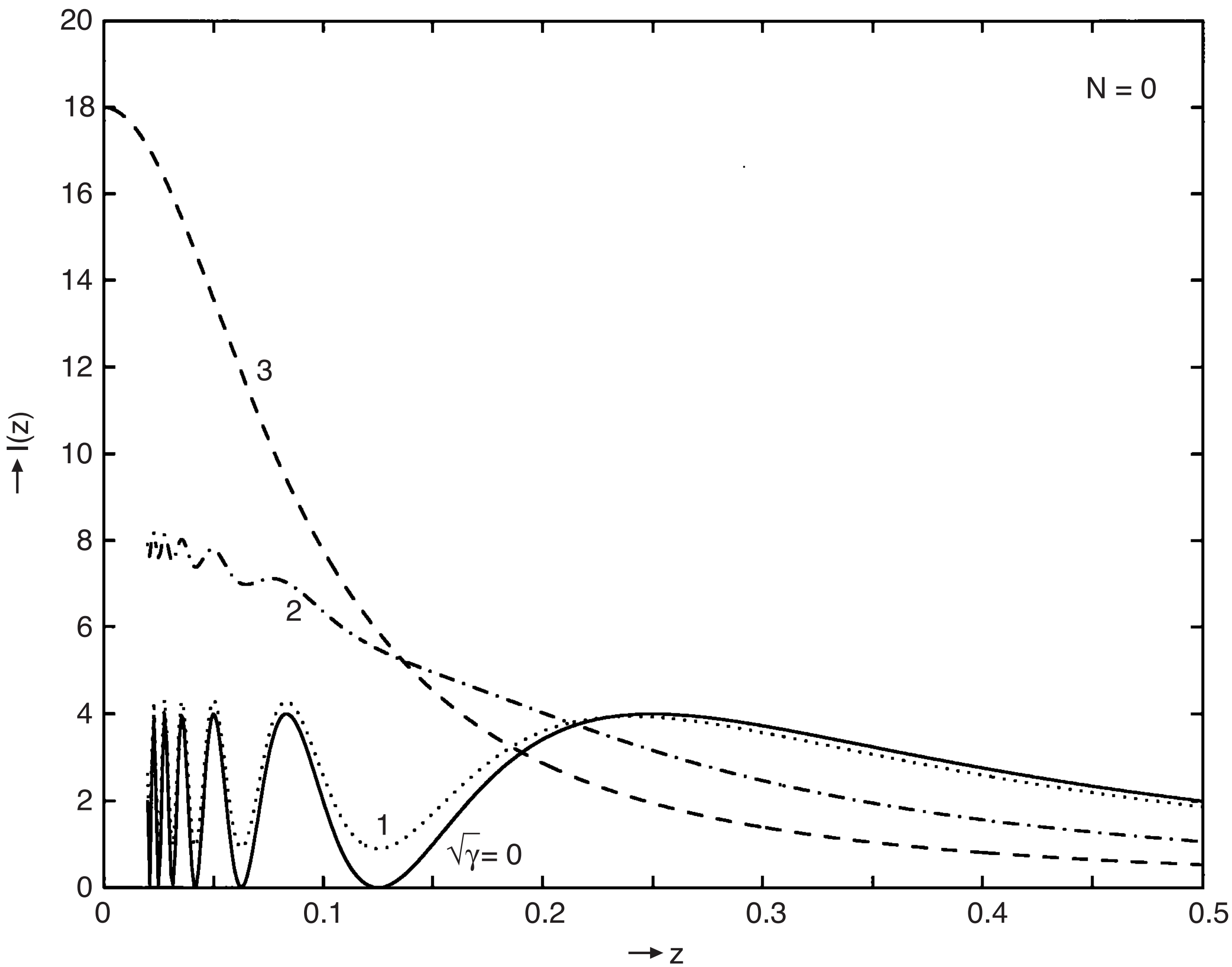
• The beam expands as a hyperbola with its asymptotes at an angle

$$\theta = \omega_z/z = \lambda/\pi\omega_0$$

called the divergence angle or the far-field diffraction angle of the beam.

• Backward propagation of the beam is also shown. For large values of z, the beam radius increases linearly with it.

• Radius of curvature of the beam wavefront has a value of infinity at $z = 0$, and a value of $-2z_R$ at the Rayleigh range z_R. At large distances $z >> z_R$, $R_z \to -z$, as for a spherical wave with its center of curvature at $z = 0$.



Axial irradiance of a collimated ($N = 0$) Gaussian beam in units of the pupil irradiance P_{ex}/S_{ex} for a uniform beam. Distance z is units of the far-field distance D^2/λ.

OTF of an image in a plane at a distance z

OTF is Fourier transform of PSF (which represents the image irradiance distribution for unity total power).

Fourier transform of the radially symmetric PSF is equal to its zero-order Hankel transform:

$$\tau(v) = 2\pi \int_0^\infty PSF(r)\, J_0(2\pi v r)\, r\, dr$$

$$PSF(r_i; z) = \frac{I_i(r_i; z)}{P_{ex}} = \frac{2}{\pi \omega_z^2} \exp\left(-\frac{2 r_i^2}{\omega_z^2}\right)$$

$$\therefore \quad \tau(v_i) = \exp\left(-\pi^2 \omega_z^2 v_i^2 / 2\right) \quad , \quad 0 \le v_i \le D/\lambda z$$

- $D/\lambda z$ *is the cutoff spatial frequency.*

OTF by autocorrelating the pupil function is considered in Home Work 14-1.

Strehl Ratio

Aberrated PSF:

$$I(r;\theta_i;\gamma) = \pi^{-2}\left|\int_0^1\int_0^{2\pi}\sqrt{I(\rho)}\exp\left[i\Phi\left(\rho,\theta_p\right)\right]\exp\left[-\pi i\rho r\cos\left(\theta_p-\theta_i\right)\right]\rho\,d\rho\,d\theta_p\right|^2$$

$$I(0;\gamma) = \pi^{-2}\left|\int_0^1\int_0^{2\pi}\sqrt{I(\rho)}\exp\left[i\Phi\left(\rho,\theta_p\right)\right]\rho\,d\rho\,d\theta_p\right|^2$$

Strehl ratio:

$$S = \left|\int_0^1\int_0^{2\pi}A(\rho)\exp[i\Phi(\rho,\theta)]\,\rho\,d\rho\,d\theta\right|^2 \Big/ \left[\int_0^1\int_0^{2\pi}A(\rho)\,\rho\,d\rho\,d\theta\right]^2$$

$$= \left\{\frac{\gamma}{\pi[1-\exp(-\gamma)]}\right\}^2\left|\int_0^1\int_0^{2\pi}\exp\left(-\gamma\rho^2\right)\exp[i\Phi(\rho,\theta)]\,\rho\,d\rho\,d\theta\right|^2$$

$$\simeq \left(\frac{\gamma}{\pi}\right)^2\left|\int_0^\infty\int_0^{2\pi}\exp\left(-\gamma\rho^2\right)\exp[i\Phi(\rho,\theta)]\,\rho\,d\rho\,d\theta\right|^2$$

For numerical calculations:

$$S = \left[\frac{\gamma}{1-\exp(-\gamma)}\right]^2 \left|\int_0^1 \exp(-\gamma x)\, f(x)\, dx\right|^2 \quad , \quad \text{where}$$

$$f(x) = \begin{cases} \exp\left[i\left(A_s x^2 + B_d x\right)\right] & \text{Spherical + defocus} \\ J_0\left(A_c x^{3/2} + B_t x^{1/2}\right) & \text{Coma + tilt} \\ \exp\left[i(0.5A_a + B_d)x\right] J_0(0.5 A_a x) & \text{Astigmatism + defocus} \end{cases}$$

Approximate expression for Strehl ratio for small aberrations:

$$S_3 \simeq \exp\left(-\sigma_\Phi^2\right) \quad , \qquad \sigma_\Phi^2 = <\Phi^2> - <\Phi>^2$$

$$<\Phi^n> = \frac{\int_0^1 \int_0^{2\pi} A(\rho)\, [\Phi(\rho,\theta)]^n\, \rho\, d\rho\, d\theta}{\int_0^1 \int_0^{2\pi} A(\rho)\, \rho\, d\rho\, d\theta} = \frac{\gamma \int_0^1 \int_0^{2\pi} \exp\left(-\gamma\rho^2\right) [\Phi(\rho,\theta)]^n\, \rho\, d\rho\, d\theta}{\pi[1-\exp(-\gamma)]}$$

$$\simeq \frac{\gamma}{\pi} \int_0^\infty \int_0^{2\pi} \exp\left(-\gamma\rho^2\right) [\Phi(\rho,\theta)]^n\, \rho\, d\rho\, d\theta$$

Table 4-3. Zernike-Gauss radial polynomials corresponding to balanced primary aberrations in Gaussian beams.

Aberration	**Radial Polynomial**	**Gaussian***	**Uniform** $\gamma = 0$	**Gaussian** $\gamma = 1$	**Weakly-Truncated Gaussian**
Piston	R_0^0	1	1	1	1
Distortion (tilt)	R_1^1	$a_1^1\rho$	ρ	1.09367ρ	$\sqrt{\gamma/2}\rho$
Field curvature (defocus)	R_2^0	$a_2^0\rho^2 + b_2^0$	$2\rho^2 - 1$	$2.04989\rho^2 - 0.85690$	$(\gamma\rho^2 - 1)/\sqrt{3}$
Astigmatism	R_2^2	$a_2^2\rho^2$	ρ^2	$1.14541\rho^2$	$(\gamma/\sqrt{6})\rho^2$
Coma	R_3^1	$a_3^1\rho^3 + b_3^1\rho$	$3\rho^3 - 2\rho$	$3.11213\rho^3 - 1\cdot 89152\rho$	$\sqrt{\gamma/2}\left(\frac{\gamma}{2}\rho^3 - \rho\right)$
Spherical aberration	R_4^0	$a_4^0\rho^4 + b_4^0\rho^2 + c_4^0$	$6\rho^4 - 6\rho^2 + 1$	$6.12902\rho^4 - 5.71948\rho^2 + 0.83368$	$(\gamma^2\rho^4 - 4\gamma\rho^2 + 2)/2\sqrt{5}$

$$*a_1^1 = (2p_2)^{-1/2}, a_2^0 = [3(p_4 - p_2^2)]^{-1/2}, b_2^0 = -\, p_2 a_2^0, a_2^2 = (3p_4)^{-1/2}, a_3^1 = \frac{1}{2}\left(p_6 - p_4^2/p_2\right)^{-1/2}, b_3^1 = -\left(p_4/p_2\right)a_3^1,$$

$$a_4^0 = \left\{5[p_8 - 2K_1 p_6 + (K_1^2 + 2K_2)p_4 - 2K_1 K_2 p_2 + K_2^2]\right\}^{-1/2}, b_4^0 = -K_1 a_4^0, c_4^0 = K_2 a_4^0,$$

$$p_s = <\rho^s> = (1 - \exp\gamma)^{-1} + (s/2\gamma)p_{s-2}, \ s \text{ is an even integer},$$

$$p_0 = 1, K_1 = \left(p_6 - p_2 p_4\right)\Big/\left(p_4 - p_2^2\right), K_2 = \left(p_2 p_6 - p_4^2\right)\Big/\left(p_4 - p_2^2\right)$$

Aberration Balancing

$$p_s = \langle \rho^s \rangle = \frac{\int_0^1 \rho^s A(\rho)\ \rho\, d\rho}{\int_0^1 A(\rho)\, \rho\, d\rho} = \frac{\int_0^1 \rho^s \exp\left(-\gamma\rho^2\right)\ \rho\, d\rho}{\int_0^1 \exp\left(-\gamma\rho^2\right) \rho\, d\rho}$$

$$= \frac{\left[\left(\rho^s / -2\gamma\right)\exp\left(-\gamma\rho^2\right)\right]_0^1}{\left[(-1/2\gamma)\exp\left(-\gamma\rho^2\right)\right]_0^1} - \frac{s\int_0^1 \left(\rho^{s-1} / -2\gamma\right)\exp\left(-\gamma\rho^2\right) d\rho}{\int_0^1 \exp\left(-\gamma\rho^2\right) \rho\, d\rho}$$

$$= \frac{\exp(-\gamma)}{\exp(-\gamma) - 1} + \frac{s}{2\gamma} \frac{\int_0^1 \rho^{s-2} \exp\left(-\gamma\rho^2\right)\ \rho\, d\rho}{\int_0^1 \exp\left(-\gamma\rho^2\right) \rho\, d\rho}$$

$$= \frac{1}{1 - \exp(\gamma)} + \frac{s}{2\gamma} \langle \rho^{s-2} \rangle$$

For large γ, as for a weakly-truncated beam:

$$p_s = <\rho^s> \simeq \frac{s}{2\gamma} <\rho^{s-2}>$$

$$= \frac{s}{2\gamma} \frac{s-2}{2\gamma} <\rho^{s-4}>$$

$$= \frac{s}{2\gamma} \frac{s-2}{2\gamma} \frac{s-4}{2\gamma} \cdots \frac{2}{2\gamma}$$

$$= \frac{1}{\gamma^{s/2}} \frac{s}{2}\left(\frac{s}{2}-1\right)\left(\frac{s}{2}-2\right)\cdots 1 = \frac{(s/2)!}{\gamma^{s/2}}$$

Substituting for p_s in the coeffcients of the terms of a radial polynomial, we obtain the radial polynomials for weakly-truncated Gaussian pupils given in Table 4-3.

• If we normalize r_p by ω (instead of by a), then γ disappears from these expressions.

Radial variable in the pupil plane normalized by the beam radius ω:

$$\rho' = \sqrt{\gamma}\,\rho = \frac{r_p}{\omega}$$

Let A_i' be the peak value of a Seidel aberration at $\rho' = 1$, i.e., at $r_p = \omega$.

• Aberration coefficients A_i' are related to the coefficients A_i according to

$$A_s' = A_s/\gamma^2,\ A_c' = A_c/\gamma^{3/2},\ A_a' = A_a/\gamma,\ B_d' = B_d/\gamma,\ B_t' = B_t/\sqrt{\gamma}$$

Why define the primed aberration coefficients?

Since the power in a weakly-truncated Gaussian beam is concentrated in a small region near the center of the pupil, the effect of the aberration in its outer region is negligible.

Accordingly, the aberration tolerances in terms of the peak value of the aberration at the edge of the pupil $(\rho = 1)$ are not very meaningful.

Point of minimum aberration variance:

$$B_d = -(4/\gamma) A_s = -4\gamma A_s'$$

$$B_t = -(2/\gamma) A_c = -2\sqrt{\gamma} A_c'$$

$$B_d = -(1/2) A_a = -(\gamma/2) A_a'$$

• The amount of balancing aberration decreases as γ increases in the case of spherical aberration and coma, but does not change in the case of astigmatism.

• In the case of spherical aberration, the amount of balancing defocus for a weakly-truncated Gaussian beam is $4/\gamma$ times the corresponding amount for a uniform beam.

• In the case of coma, the balancing tilt for a weakly-truncated Gaussian beam is $3/\gamma$ times the corresponding amount for a uniform beam.

For large γ, the aberration tolerance in terms of the coefficients A_i' is given in Table 4-4 for a Strehl ratio of 0.8.

Table 4-4. Primary aberrations and their standard deviations for weakly-truncated ($\sqrt{\gamma} \geq 3$) Gaussian pupils.

Aberration	$\boldsymbol{\Phi(\rho',\theta)}$	$\boldsymbol{\sigma_\Phi}$	$\boldsymbol{A_i'}$ **for** $\boldsymbol{S = 0.8}$
Spherical	$A_s' \rho'^4$	$2\sqrt{5}A_s'$	$\lambda/63$
Balanced spherical	$A_s'\left(\rho'^4 - 4\rho'^2\right)$	$2\,A_s'$	$\lambda/28$
Coma	$A_c' \rho'^3 \cos\theta$	$\sqrt{3}A_c'$	$\lambda/24$
Balanced coma	$A_c'\left(\rho'^3 - 2\rho'\right)\cos\theta$	A_c'	$\lambda/14$
Astigmatism	$A_a' \rho'^2 \cos^2\theta$	$A_a'/\sqrt{2}$	$\lambda/10$
Balanced astigmatism	$A_a' \rho'^2 \left(\cos^2\theta - 1/2\right)$	$A_a'/2$	$\lambda/7$
Defocus	$B_d' \rho'^2$	$\sqrt{3}B_d'$	$\lambda/24$
Tilt	$B_t' \rho' \cos\theta$	$\sqrt{3}B_t'$	$\lambda/20$

Table. Primary aberrations and their standard deviations.

Seidel Aberration	$\sigma_\Phi(\gamma = 0)$	$\sigma_\Phi(\gamma = 1)$	$\sigma_\Phi(\sqrt{\gamma} = 2)$	$\sigma_\Phi(\sqrt{\gamma} \geq 3)$
Spherical, $A_s \rho^4$	$\frac{2A_s}{3\sqrt{5}} = \frac{A_s}{3.35}$	$\frac{A_s}{3.67}$	$\frac{A_s}{6.20}$	$\frac{2\sqrt{5}A_s}{\gamma^2}$
Coma, $A_c \rho^3 \cos\theta$	$\frac{A_c}{2\sqrt{2}} = \frac{A_c}{2.83}$	$\frac{A_c}{3.33}$	$\frac{A_c}{6.08}$	$\frac{\sqrt{3}A_c}{\gamma^{3/2}}$
Astigmatism, $A_a \rho^2 \cos^2\theta$	$\frac{A_a}{4}$	$\frac{A_a}{4.40}$	$\frac{A_a}{6.59}$	$\frac{A_a}{\sqrt{2}\gamma}$
Defocus, $B_d \rho^2$	$\frac{B_d}{2\sqrt{3}} = \frac{B_d}{3.46}$	$\frac{B_d}{3.55}$	$\frac{B_d}{4.79}$	$\frac{B_d}{\gamma}$
Tilt, $B_t \rho \cos\theta$	$\frac{B_t}{2}$	$\frac{B_t}{2.19}$	$\frac{B_t}{2.94}$	$\frac{B_t}{\sqrt{2}\gamma}$

- Sigma decreases as γ increases. Hence, for small aberrations, the aberration tolerance increases.

Table. Balanced primary aberrations.

Balanced Aberration	$\boldsymbol{\Phi(\rho, \theta; \gamma = 0)}$	$\boldsymbol{\Phi(\rho, \theta; \gamma = 1)}$	$\boldsymbol{\Phi(\rho, \theta; \sqrt{\gamma} = 2)}$	$\boldsymbol{\Phi(\rho, \theta; \sqrt{\gamma} \geq 3)}$
Spherical	$A_s\left(\rho^4 - \rho^2\right)$	$A_s\left(\rho^4 - 0.933\rho^2\right)$	$A_s\left(\rho^4 - 0.728\rho^2\right)$	$A_s\left(\rho^4 - \frac{4}{\gamma}\rho^2\right)$
Coma	$A_c\left(\rho^3 - \frac{2}{3}\rho\right)\cos\theta$	$A_c\left(\rho^3 - 0.608\rho\right)\cos\theta$	$A_c\left(\rho^3 - 0.419\rho\right)\cos\theta$	$A_c\left(\rho^3 - \frac{2}{\gamma}\rho\right)\cos\theta$
Astigmatism	$A_a\rho^2\left(\cos^2\theta - 1/2\right)$	$A_a\rho^2\left(\cos^2\theta - 1/2\right)$	$A_a\rho^2\left(\cos^2\theta - 1/2\right)$	$A_a\rho^2\left(\cos^2\theta - 1/2\right)$

As γ increases:

- Balancing defocus decreases for spherical aberration.
- Balancing tilt decreases for coma.
- Balancing defocus for astigmatism does not change.

Table. Diffraction focus.

Balanced Aberration	Diffraction Focus			
	Uniform $(\boldsymbol{\gamma} = \mathbf{0})$	**Gaussian** $(\boldsymbol{\gamma} = \mathbf{1})$	**Gaussian** $(\sqrt{\boldsymbol{\gamma}} = \mathbf{2})$	**Weakly-Truncated Gaussian** $(\sqrt{\boldsymbol{\gamma}} \geq \mathbf{3})$
Spherical	$\left(0,0,8F^2A_s\right)$	$\left(0,0,7.46F^2A_s\right)$	$\left(0,0,5.82F^2A_s\right)$	$\left(0,0,\dfrac{32}{\gamma}F^2A_s\right)$
Coma	$\left(4FA_c/3,0,0\right)$	$\left(1.22FA_c,0,0\right)$	$\left(0.84FA_c,0,0\right)$	$\left(4FA_c/\gamma,0,0\right)$
Astigmatism	$\left(0,0,4F^2A_a\right)$	$\left(0,0,4F^2A_a\right)$	$\left(0,0,4F^2A_a\right)$	$\left(0,0,4F^2A_a\right)$

As γ increases:

- Diffraction focus for spherical aberration moves closer to the focal point.
- Diffraction focus for coma moves closer to the focal point.
- Diffraction focus for astigmatism does not change.

Table. Standard deviation of balanced primary aberrations.

Balanced Aberration	$\sigma_\Phi(\gamma = 0)$	$\sigma_\Phi(\gamma = 1)$	$\sigma_\Phi(\sqrt{\gamma} = 2)$	$\sigma_\Phi(\sqrt{\gamma} \geq 3)$
Spherical	$\frac{A_s}{6\sqrt{5}} = \frac{A_s}{13.42}$	$\frac{A_s}{13.71}$	$\frac{A_s}{18.29}$	$\frac{2\,A_s}{\gamma^2}$
Coma	$\frac{A_c}{6\sqrt{2}} = \frac{A_c}{8.49}$	$\frac{A_c}{8.80}$	$\frac{A_c}{12.21}$	$\frac{A_c}{\gamma^{3/2}}$
Astigmatism	$\frac{A_a}{2\sqrt{6}} = \frac{A_a}{4.90}$	$\frac{A_a}{5.61}$	$\frac{A_a}{9.08}$	$\frac{A_a}{2\gamma}$

• Sigma decreases as γ increases. Hence, for small aberrations, the aberration tolerance increases.

Table. Factor by which the standard deviation of a Seidel aberration is reduced when it is optimally balanced with other aberrations.

	Reduction Factor			
Balanced Aberration	**Uniform** $(\gamma = 0)$	**Gaussian** $(\gamma = 1)$	**Gaussian** $(\sqrt{\gamma} = 2)$	**Weakly-Truncated Gaussian** $(\sqrt{\gamma} \geq 3)$
Spherical	4	3.74	2.95	$\sqrt{5}$ = 2.24
Coma	3	2.64	2.01	$\sqrt{3}$ = 1.73
Astigmatism	1.22	1.27	1.38	$\sqrt{2}$ = 1.41

• As γ increases, the reduction factor decreases for spherical aberration and coma, but increases for astigmatism.

Table 4-6. Reduction factors by which the peak aberration coefficient A_i of a primary aberration is divided to obtain the standard deviation of the corresponding balanced aberration.

$\sqrt{\gamma}$	**Balanced Spherical**	**Balanced Coma**	**Balanced Astigmatism**
0	13.42	8.49	4.90
0.5	13.69	8.53	5.06
1.0	13.71	8.80	5.61
1.5	14.90	9.74	6.81
2.0	18.29	12.21	9.08
2.5	26.33	17.62	12.82
3.0	43.52	27.57	18.06
3.5	75.78	42.96	24.51
4.0	128.09	64.01	32.00

Reduction factor:	spherical	coma	astimatism
general:	$\sqrt{5}a_4^0$	$2\sqrt{2}a_3^1$	$2\sqrt{6}a_2^2$
weakly truncated:	$\gamma^2/2$	$\gamma^{3/2}$	2γ

• Aberration-free PSF for $\sqrt{\gamma} \geq 2$ is approximately the same as that for a weakly-truncated Gaussian beam.

• However, the standard deviation of an aberration for $\sqrt{\gamma} = 2$ according to the weakly-truncated beam assumption is significantly different from its true value.

• Reason for the discrepancy is simple. Even though the irradiance in the region of the pupil $\omega/a \leq \rho \leq 1$ is quite small compared to its value at or near the center, the amplitude is not as small. Moreover, the aberration in this region can be quite large, and thus have a significant effect on the σ.

• Spherical aberration increases as ρ^4 and coma increases as ρ^3. Hence, we require a larger value of γ, namely $\sqrt{\gamma} \geq 3$, for the validity of the weakly-truncated assumption. This is also somewhat true for astigmatism and defocus, which increase as ρ^2.

Table. Standard deviation of balanced primary aberrations.*

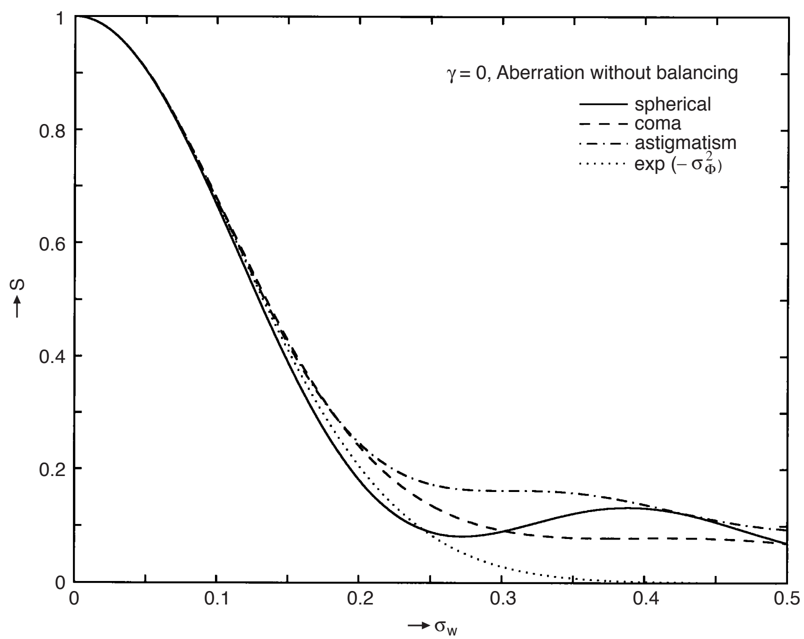
Balanced Aberration	$\sigma_\Phi(\sqrt{\gamma}=2)$ (exact)	$\sigma_\Phi(\sqrt{\gamma}=2)$ (approximate)	$\sigma_\Phi(\sqrt{\gamma}=3)$ (exact)	$\sigma_\Phi(\sqrt{\gamma}=3)$ (approximate)
Spherical	$\frac{A_s}{18.29}$	$\frac{A_s}{8}$	$\frac{A_s}{43.52}$	$\frac{A_s}{40.50}$
Coma	$\frac{A_c}{12.21}$	$\frac{A_c}{8}$	$\frac{A_c}{27.57}$	$\frac{A_c}{27}$
Astigmatism	$\frac{A_a}{9.08}$	$\frac{A_a}{8}$	$\frac{A_a}{18.06}$	$\frac{A_a}{18}$
Defocus	$\frac{B_d}{4}$	$\frac{B_d}{4.80}$	$\frac{B_d}{9.05}$	$\frac{B_d}{9}$

* Approximate results are obtained under the weakly-truncated assumption.

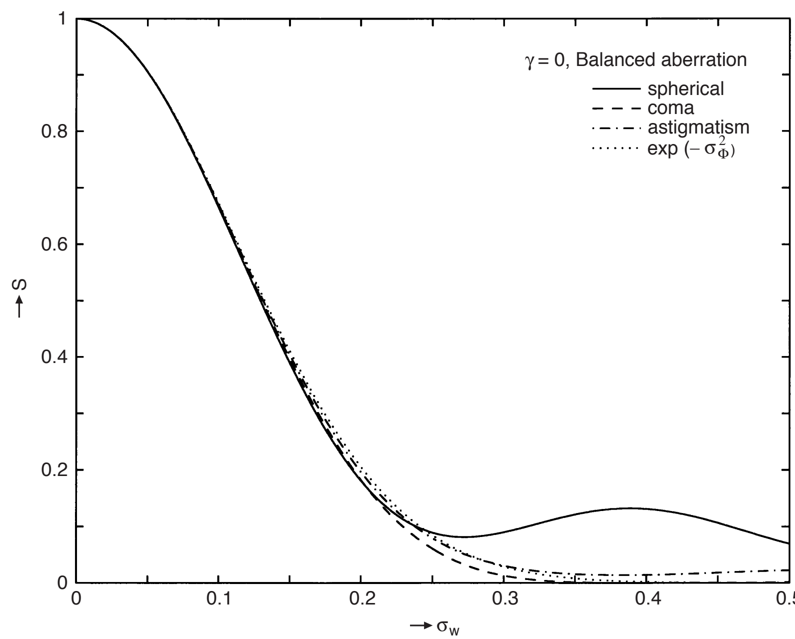
Table 11. Strehl ratio for primary aberrations.

Aberration	$S(\gamma = 0)$	$S(\sqrt{\gamma} \geq 3)$
Spherical, $A_s\rho^4$	$\frac{1}{b}\left\{C^2(\sqrt{b}) + S^2(\sqrt{b})\right\}$	$\frac{b'^2}{\pi}\left\{\left[\frac{1}{2} - S\left(\frac{1}{\sqrt{b'}}\right)\right]^2 + \left[\frac{1}{2} - C\left(\frac{1}{\sqrt{b'}}\right)\right]^2\right\}$
Balanced spherical, $A_s\rho^4 + B_d\rho^2$	$\frac{1}{b}\left\{C^2(\sqrt{b}) + S^2(\sqrt{b})\right\}$	Numerical integration with $B_d = -(4/\gamma)A_s$ and $A_s = \gamma^2\sigma_s/2$
Coma, $A_c\rho^3\cos\theta$	$\left[\int_0^1 J_0\left(2\sqrt{2}\sigma_c x^{3/2}\right)dx\right]^2$	Numerical integration with $B_t = 0$ and $A_c = \gamma^{3/2}\sigma_c/3$
Balanced coma, $\left(A_c\rho^3 + B_t\rho\right)\cos\theta$	$\left[\int_0^1 J_0\left[2\sqrt{6}\sigma_{bc}\left(x^{3/2} - \frac{2}{3}x^{1/2}\right)\right]dx\right]^2$	Numerical integration with $B_t = -(2/\gamma)A_c$ and $A_c = \gamma^{3/2}\sigma_{bc}$
Astigmatism, $A_a\rho^2\cos^2\theta$	$J_0^2(2\sigma_a) + J_1^2(2\sigma_a)$	$\left[1 + (A_a/\gamma)^2\right]^{-1/2} = \left[1 + 2\sigma_a^2\right]^{-1/2}$
Balanced astigmatism, $A_a\rho^2\left(\cos^2\theta - 1/2\right)$	$\frac{2}{3\sigma_{ba}^2}\left[\sum_{k=0}^{\infty} J_{2k+1}\left(\sqrt{6}\sigma_{ba}\right)\right]^2$	$\left[1 + (A_a/2\gamma)^2\right]^{-1} = \frac{1}{1 + \sigma_{ba}^2}$
Defocus, $B_d\rho^2$	$\left(\frac{\sin\sqrt{3}\sigma_d}{\sqrt{3}\sigma_d}\right)^2$	$\left[1 + (B_d/\gamma)^2\right]^{-1} = \frac{1}{1 + \sigma_d^2}$

* $b = 3\sqrt{5}\sigma_s/\pi$, $b' = \pi\sigma_s$, $C(b) = \int_0^b \cos\left(\pi x^2/2\right)dx$, $S(b) = \int_0^b \sin\left(\pi x^2/2\right)dx$



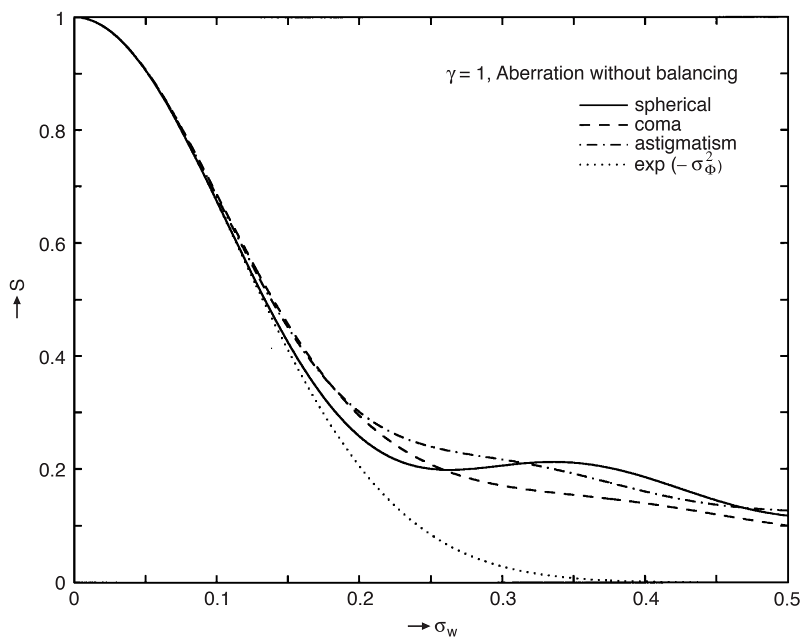
(a)



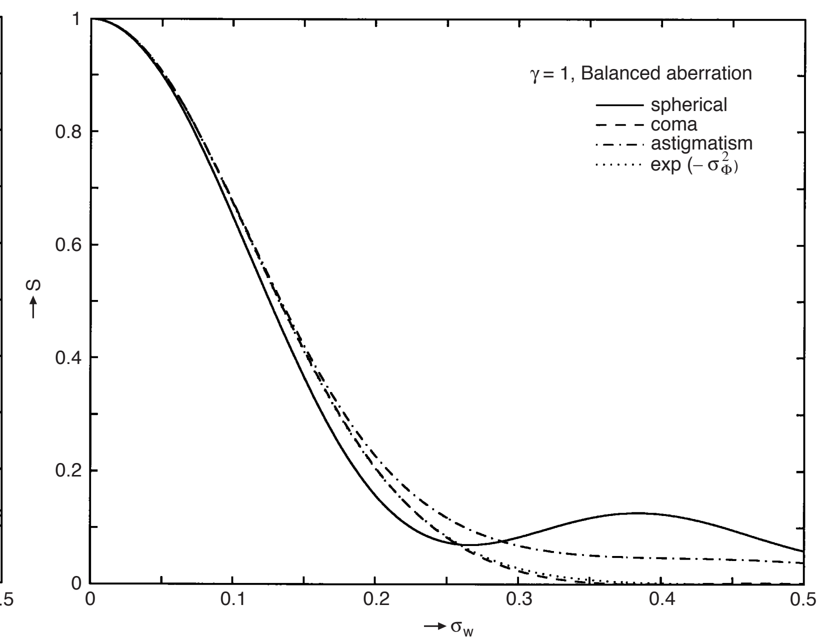
(b)

Strehl ratio of a uniform ($\gamma = 0$) beam aberrated by a primary aberration with and without balancing as a function of its standard deviation σ_w in units of wavelength λ.

- $\exp\left(-\sigma_\Phi^2\right)$ gives a reasonable approximation of Strehl ratio for $\sigma_w \lesssim \lambda/5$
- Approximation is even better for a balanced aberration and up to $\sigma_w \lesssim \lambda/4$.



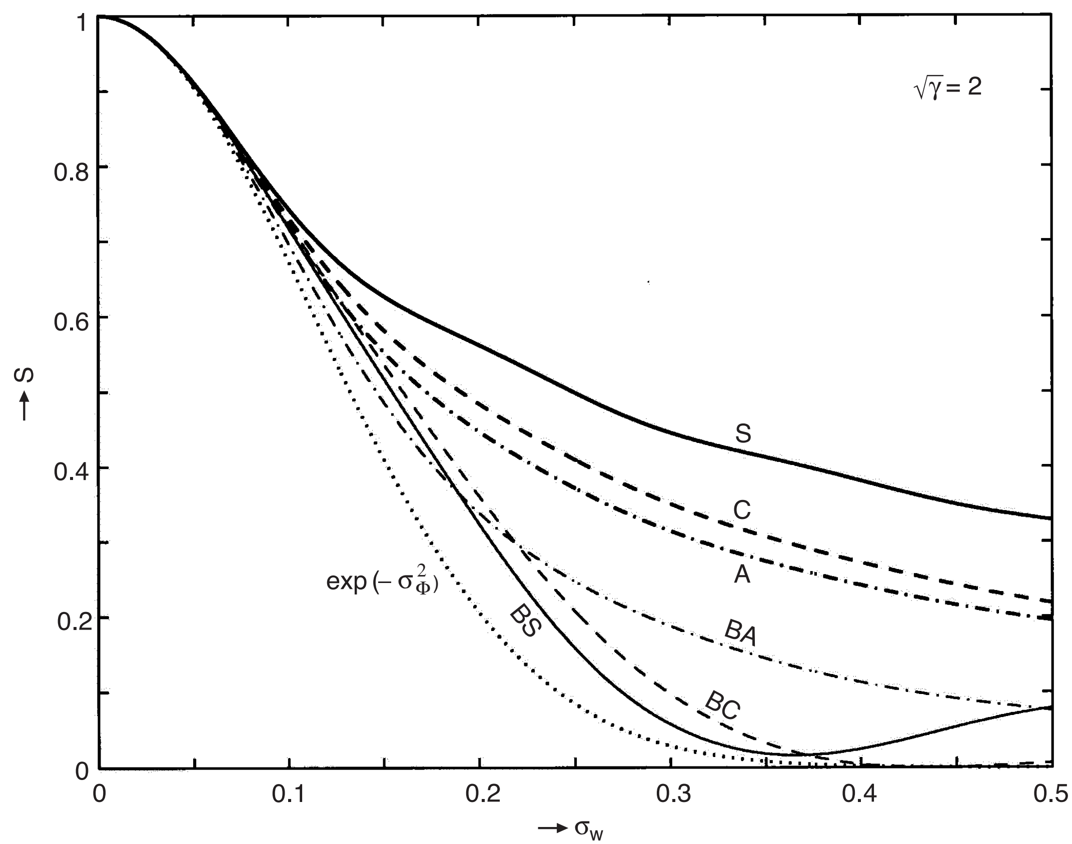
(a)



(b)

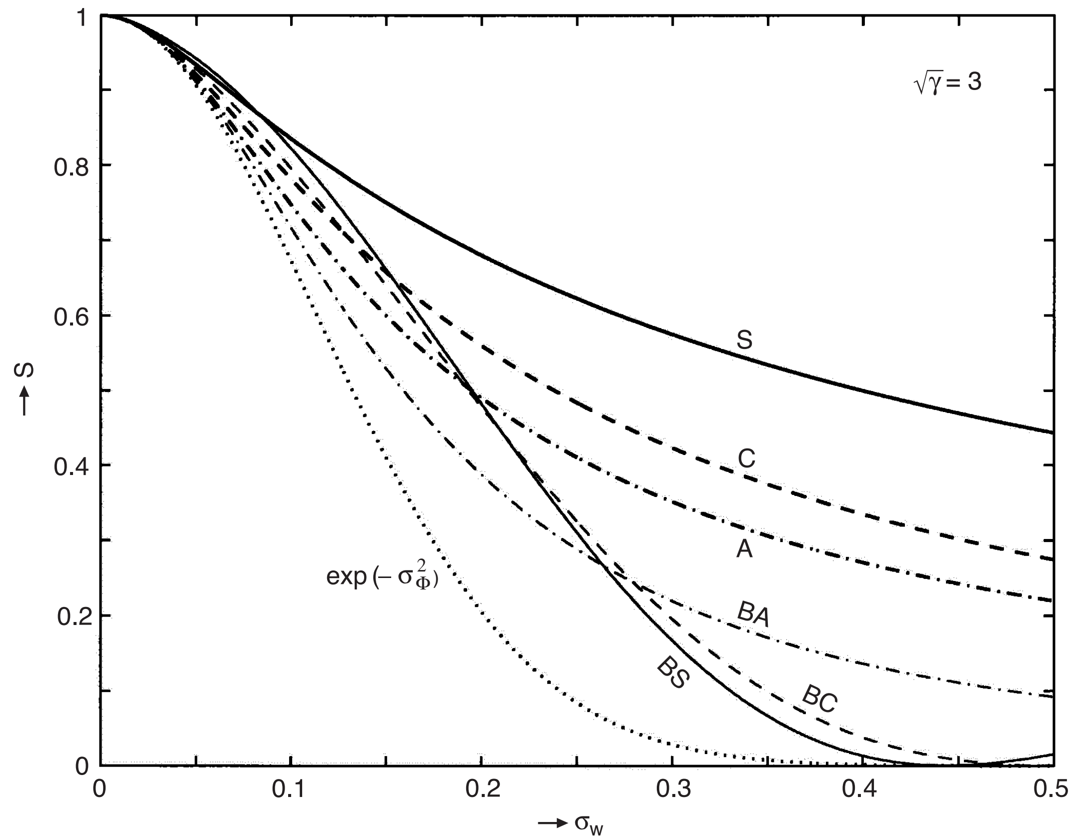
Strehl ratio of a Gaussian beam with $\gamma = 1$ aberrated by a primary aberration with and without balancing as a function of its standard deviation σ_w in units of wavelength λ.

• The range of σ_w for which $\exp\left(-\sigma_\Phi^2\right)$ gives a reasonable approximation of the Strehl ratio decreases somewhat.



Strehl ratio of a Gaussian beam with $\sqrt{\gamma} = 2$ aberrated by a primary aberration with and without balancing as a function of its standard deviation σ_w in units of wavelength λ.

• The range of σ_w for which $\exp\left(-\sigma_\Phi^2\right)$ gives a reasonable approximation of the Strehl ratio decreases as γ increases.



Strehl ratio of a weakly-truncated Gaussian beam with $\sqrt{\gamma} = 3$ aberrated by a primary aberration with and without balancing as a function of its standard deviation σ_w in units of wavelength λ.

- $\exp\left(-\sigma_\Phi^2\right)$ *gives a reasonable approximation only for very high Strehl ratios,* $S \gtrsim 0.9$*, or very small value of standard deviation,* $\sigma_w \lesssim \lambda/20$.

Line of Sight

Centroid of a PSF:

$$\langle x_i, y_i \rangle = P_{ex}^{-1} \iint (x_i, y_i) I_i(x_i, y_i)\, dx_i\, dy_i$$

- Centroid is affected by only those aberrations that vary as $\cos\theta$.
- Centroid of a PSF aberrated by coma $A_c \rho^3 \cos\theta$:

$$\langle x_i \rangle = 4 A_c F \left[\frac{1}{2\gamma} + \frac{1}{1 - \exp(2\gamma)} \right]$$

- $\langle y_i \rangle = 0$ since the PSF is symmetric about the x_i axis.
- $R_3^1(\rho;\gamma)$ yields the point x_m in the image plane with respect to which the aberration variance is minimum:

$$x_m = 2 A_c F \left[\frac{2}{\gamma} + \frac{\gamma}{1 + \gamma - \exp(\gamma)} \right]$$

Uniform pupil ($\gamma = 0$):

$$\langle x_i \rangle = 2 A_c F \quad \text{and} \quad x_m = (4/3) A_c F$$

Table 10. Minimum variance point x_m, peak value point x_p, and the centroid $<x_i>$ in units of λF, and the corresponding irradiances I_m, I_p, and I_c for a Gaussian pupil with $\gamma = 1$ aberrated by primary coma.

A_c	x_m	x_p	$<x>$	I_m	I_p	I_c	$I(0)$
0	0	0	0	1	1	1	1
	(0)	(0)	(0)	(1)	(1)	(1)	(1)
0.50	0.61	0.60	0.69	0.8805	0.8806	0.8670	0.4567
	(0.67)	(0.67)	(1)	(0.8712)	(0.8712)	(0.6535)	(0.3175)
1.00	1.22	1.15	1.37	0.6013	0.6062	0.5590	0.1708
	(1.33)	(1.30)	(2)	(0.5708)	(0.5717)	(0.1445)	(0.0791)
1.50	1.82	1.40	2.06	0.3205	0.3672	0.2479	0.1199
	(2.00)	(1.80)	(3.00)	(0.2715)	(0.2844)	(0.0004)	(0.0618)
2.00	2.43	1.46	2.75	0.1305	0.2947	0.0624	0.0733
	(2.67)	(1.57)	(4.00)	(0.0864)	(0.1978)	(0.0061)	(0.0341)

• For small values of A_c, the peak value I_p of the aberrated PSF occurs at a point x_p that is (approximately) equal to x_m.

• Value of x is smaller and the corresponding irradiance is larger for a Gaussian pupil than that for a uniform pupil (given in parentheses).

Table. Comparison of uniform and Gaussian beams

Characteristic	Uniform Beam	Gaussian beam
Focal-point irradiance	Larger	Smaller
Encircled power	Higher for small circles	Higher for large circles
Central bright spot and rings	Smaller bright spot with more power in rings	Larger bright spot with less power in rings
OTF	Higher at high frequencies	Higher at low frequencies
Beam propagation	Complex structure	Complex structure simplifies for a weak truncation---remains Gaussian
Axial irradiance	Zero minima, which disappear	Nonzero minima, which disappear for weak truncation
Depth of focus	Small, unless the Fresnel number is very small	Increases as γ increases
Aberration Tolerance	Less	More, increases as γ increases
Validity of weak truncation	Not applicable	$a > 2\omega$ for aberration free, but $a > 3\omega$ for an aberrated beam
Approximate Strehl ratio $\exp\left(-\sigma_\Phi^2\right)$	Good for $S \geq 0.3$	Range of validity decreases as γ increases; $S \geq 0.9$ for a weakly truncated beam
Centroid	Same for primary and secondary coma of the same peak value	Different for primary and secondary coma of the same peak value

Home Work 14

14-1. (a) Using the autocorrelation integral, show that the OTF of a weakly-truncated Gaussian pupil is given by

$$\tau(v;\gamma) = \exp\left\{-\left[2\gamma+(2/\gamma)B_d^2\right]v^2\right\} \quad , \quad 0 \le v \le 1$$

where the spatial frequency v is normalized by the cutoff frequency $D/\lambda z$.

(b) Show that it reduces to the form

$$\tau(v_i) = \exp\left(-\pi^2\omega_z^2 v_i^2/2\right) \quad , \quad 0 \le v_i \le D/\lambda z$$

obtained by Fourier transforming the PSF, where ω_z is the beam radius at a distance z given by $\omega_z^2 = (\lambda z/\pi\omega)^2 + \omega^2(1-z/R)^2$.

14-2. (a) Consider a Gaussian pupil with $\sqrt{\gamma} = 2$. Determine the standard deviation of a primary aberration, and the corresponding balanced primary aberration and its standard deviation. (b) Also, determine the form of the Zernike radial polynomials for primary aberrations.