

Lecture 6

Mirror Matrices

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Law of Reflection

Matrix formalism is used to model reflection from plane mirrors.
Start with the vector law of reflection:

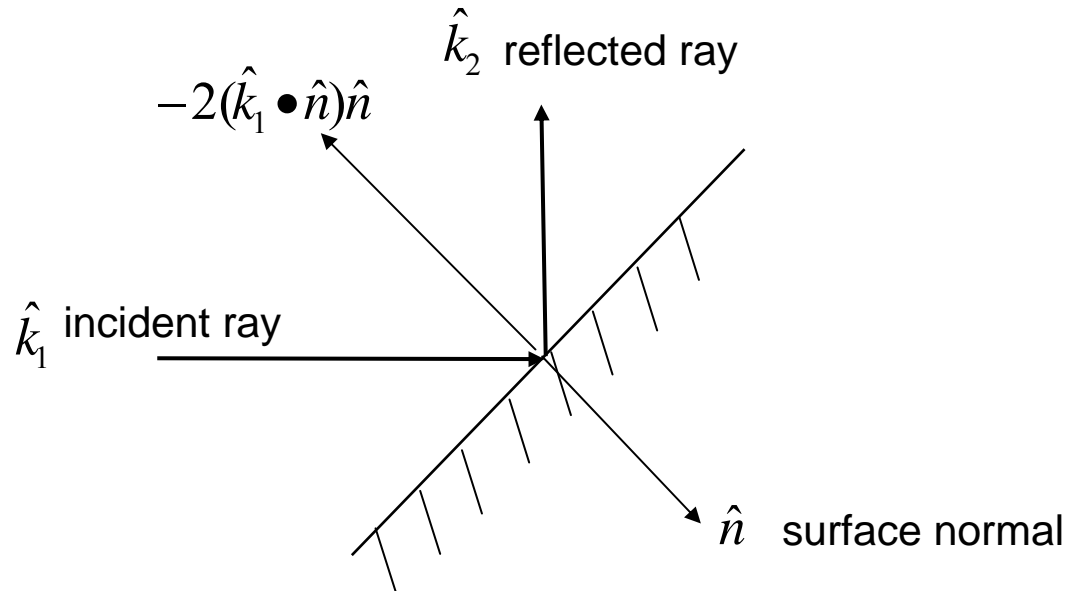
$$\hat{k}_2 = \hat{k}_1 - 2(\hat{k}_1 \cdot \hat{n})\hat{n}$$

The hats indicate unit vectors

\hat{k}_1 = incident ray

\hat{k}_2 = reflected ray

\hat{n} = surface normal



For a plane mirror with its normal vector \mathbf{n} with (x,y,z) components (n_x, n_y, n_z)
Using the standard vector representation with unit vectors,

$$\hat{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k}$$

The matrix representation of this vector is

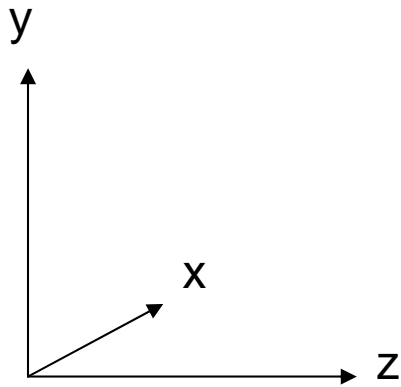
$$\mathbf{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

The vector law of reflection can be written in matrix form as

$$k_2 = M k_1$$

Where the mirror matrix M is calculated to be

$$M = I - 2n \cdot n^T$$



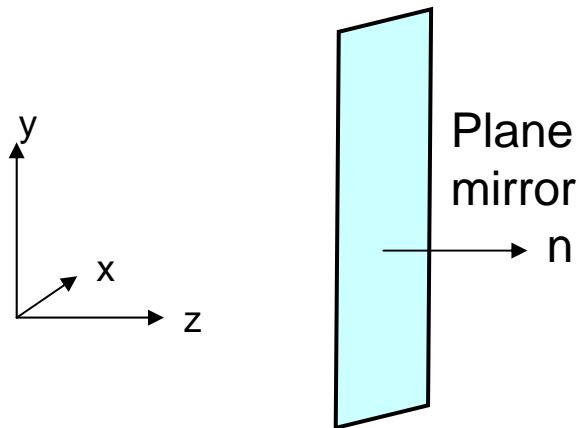
$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$k_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \begin{bmatrix} n_x & n_y & n_z \end{bmatrix}$$

Resulting mirror matrix:

$$M := \begin{bmatrix} 1 - 2 \cdot n_x^2 & -2 \cdot n_x \cdot n_y & -2 \cdot n_x \cdot n_z \\ -2 \cdot n_x \cdot n_y & 1 - 2 \cdot n_y^2 & -2 \cdot n_y \cdot n_z \\ -2 \cdot n_x \cdot n_z & -2 \cdot n_y \cdot n_z & 1 - 2 \cdot n_z^2 \end{bmatrix}$$

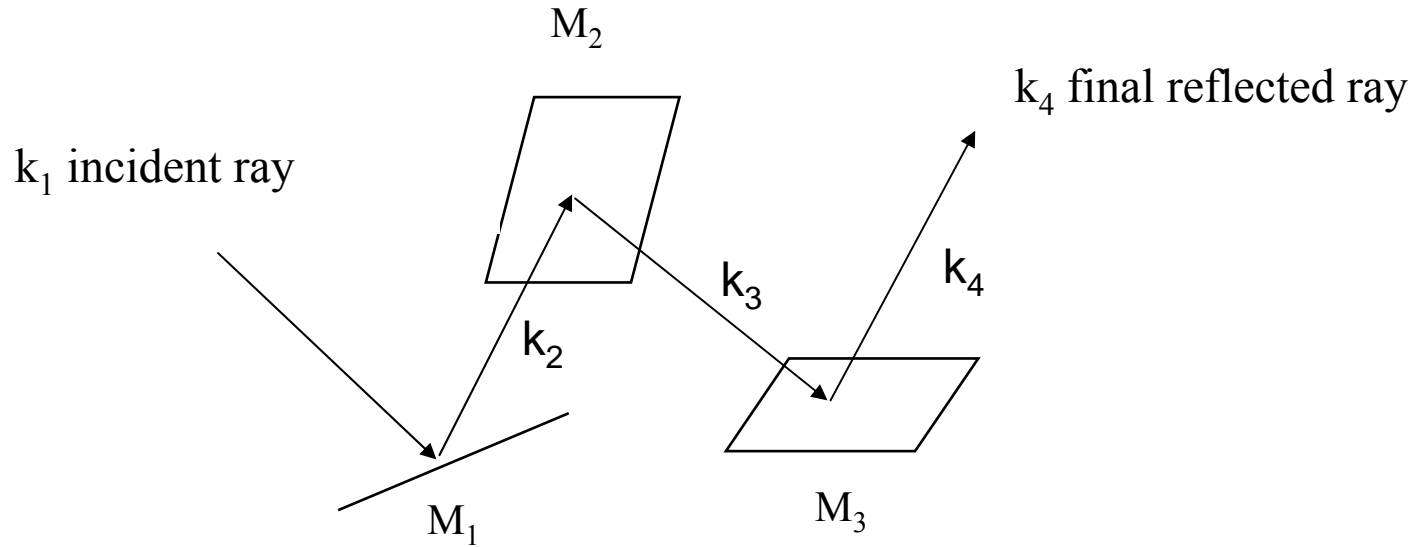
M tells you only
direction of ray,
not its position



$$n = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$n_x = 0, n_y = 0$ and $n_z = 1$ so, $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Multiple reflections



$$k_2 = M_1 k_1$$

$$k_3 = M_2 k_2 = M_2 (M_1 k_1)$$

$$k_4 = M_3 k_3 = M_3 (M_2 (M_1 k_1))$$

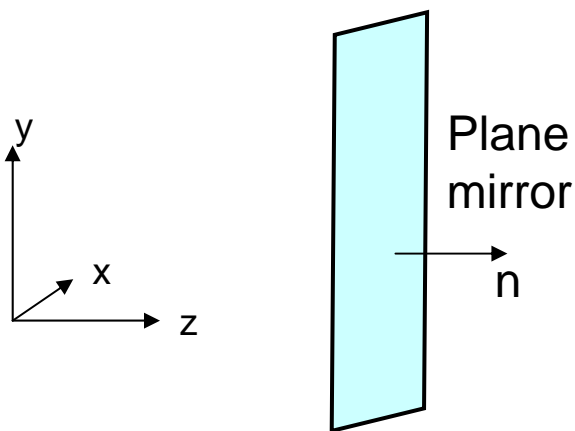
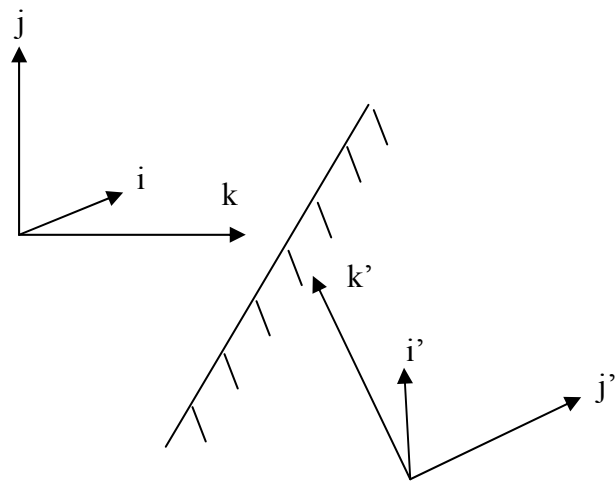
So the effect of any set of mirrors can be reduced to a single 3x3 matrix.

$$k_4 = M_3 M_2 M_1 k_1$$

$$k_4 = M_{\text{eff}} k_1$$

$$M_{\text{eff}} := M_3 \cdot M_2 \cdot M_1$$

The mirror matrix shows the reflected coordinates, not just the incident ray. Initial coordinates (i,j,k) get reflected to a new set (i',j',k')



$$M_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$x' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$y' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$z' = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Parity

- The parity of this one mirror is of course odd (-1). The image of a right handed coordinate system will appear to be left handed in the reflection.
- This means that clockwise rotation about any basis vector will appear counter-clockwise in the image.

In general, the determinant of the mirror matrix gives the parity of the system.

- An even number of reflections will cause the image to be right-handed, or to have parity = $\det(M) = 1$
- A system with an odd number of reflections will cause the image to be left-handed, or to have parity = $\det(M) = -1$

Mirror rotation

$$M_r = R \cdot M \cdot R^T$$

(M is initial mirror matrix)

where M_r is the new matrix and R is the rotation matrix given below:

x rotation: $R_x := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$

y rotation: $R_y := \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$

z rotation: $R_z := \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Transpose operation, swap rows with columns:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Mirror rotation

For the simple case of the mirror starting with its normal in the z direction, rotation α about the x axis gives

$$M_r = R_x(\alpha) M_z R_x(\alpha)^T$$

We can show that,

$$M_r = R_x(2\alpha) M_z$$

Likewise the effect of y-rotation is

$$M_r = R_y(2\beta) M_z$$

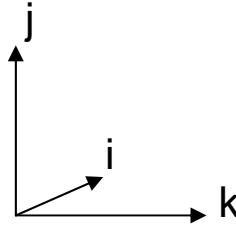
The effect of z-rotation is NOT the same. You can use trig identities to show that

$$M_r = R_z(\gamma) M_z R_z^T(\gamma) = M_z$$

Some common types of mirrors

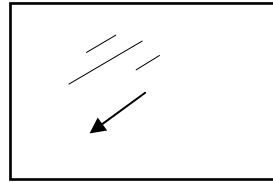
Free space:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



x mirror:

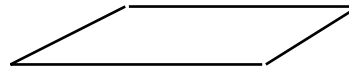
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



insensitive to x rotation
 2θ for y and z rotations

y mirror:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



insensitive to y rotation
 2θ for x and z rotations

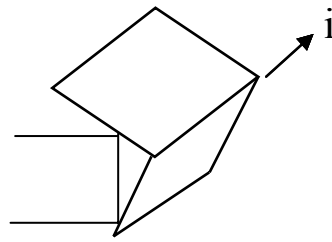
z mirror:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



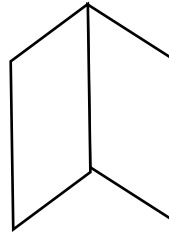
insensitive to z rotation
 2θ for x and y rotations

90° x roof:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



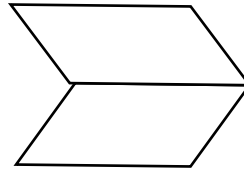
insensitive to x rotation
 2θ for y and z rotations

90° y roof:
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



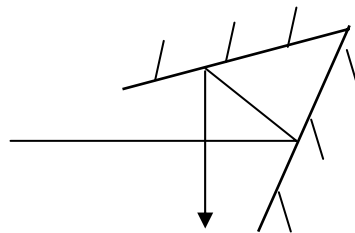
insensitive to y rotation
 2θ for x and z rotations

90° z roof:
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



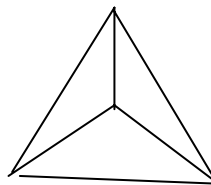
insensitive to z rotation
 2θ for x and y rotations

45° x roof:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$



90° deviation
 insensitive to x rotation
 θ for y and z rotations

cube corner:
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



retro-reflects
 insensitive to all rotations