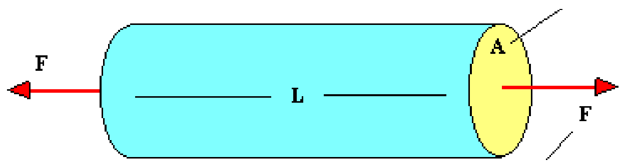
# Deflections under loading

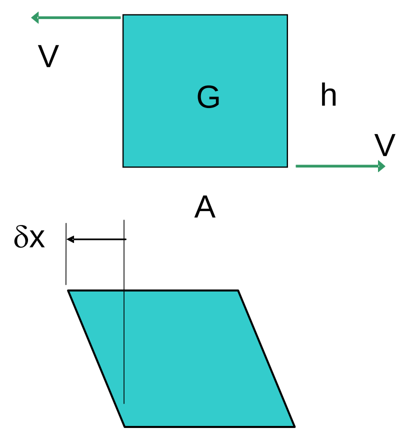
We have already seen the effects of loading for simple cases:

**Axial load F**




$$\varepsilon E = \sigma \quad \textit{giving} \quad \Delta L = \frac{FL}{EA} \quad \text{or in general} \quad \frac{dL(z)}{dz} = \frac{F}{EA}$$

**Shear load V**

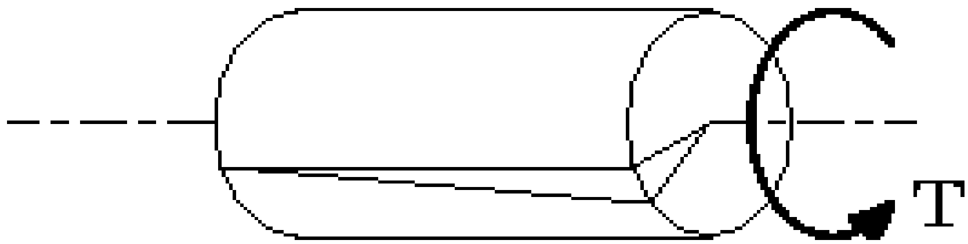


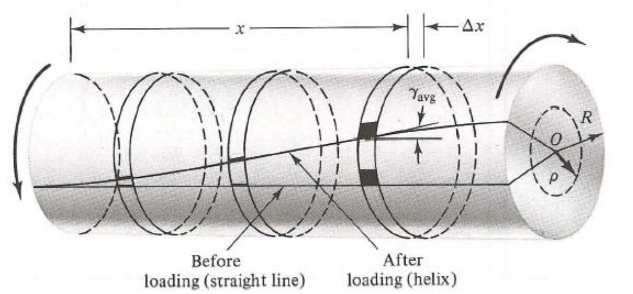

$$\gamma G = \tau \quad \text{which gives} \quad \delta x = \frac{Vh}{GA}$$

**Applying a force will cause a displacement**

# Torsion

Twisting motion about an axis

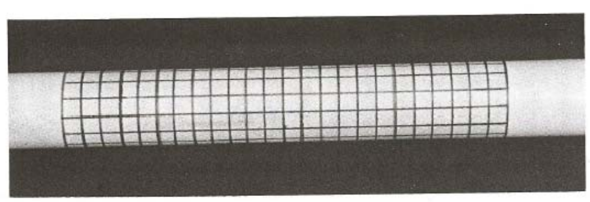


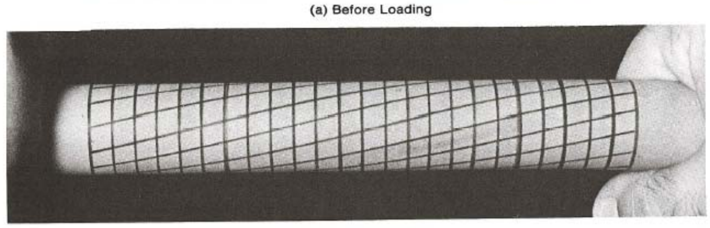

In the above example, each disk rotates slightly with respect to the previous disk and plane sections remain plane. The rotation of each disk with respect to the previous disk causes the shaded element that was originally square to change shape.

When a bar or shaft of circular cross section is loaded in this manner (*twisted by couples*) the bar is said to be in **pure torsion** and the deformed element, shown above, are said to be in a state of **pure shear**.

As shown below, lines parallel with the axis of the shaft distort into helixes. Lines perpendicular to the axis of the shaft remain perpendicular. Therefore, plane sections remain plane and undistorted.



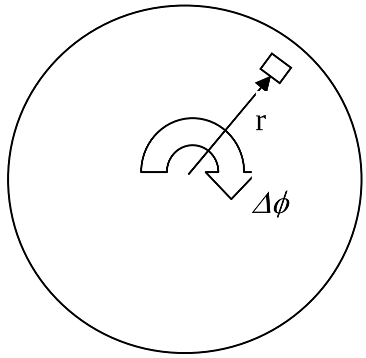
(a) Before Loading



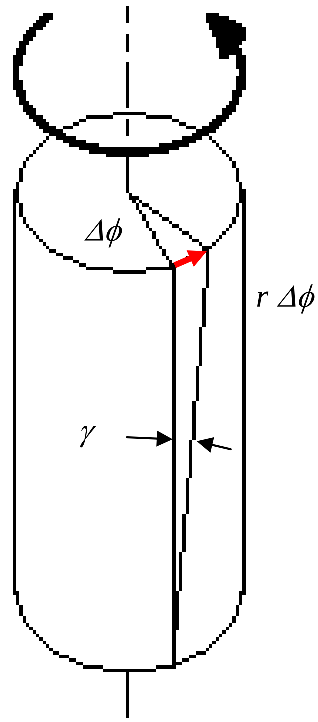
(b) After Loading

The above assumptions are correct for circular bars or shaft whether solid or hollow (i.e. round tubes) but are incorrect for other shapes. Other shapes, such as a rectangular bar, distort when twisted (*i.e. plane sections do not remain plane*).

**Applying a twisting moment will cause a rotation ϕ**




**Each element acts in shear, exerting a moment**




$$dM = r \times \tau(r) dA$$

**From elasticity**

$$\tau(r) = \gamma(r) \cdot G$$

For volume element with length L

$$r\Delta\phi = \gamma(r) \cdot L$$
$$= \frac{\tau(r)}{G} L$$

Shear deflection at $r$
Twist angle $\Delta\phi$
Total shear is $r\ \Delta\phi$

Torque T applied to the shaft

$$T = \int dM = \int r\tau(r)dA = \int r\left(\frac{r\Delta\phi}{L}G\right)dA$$

$$T = \frac{\Delta\phi}{L}G\int r^2 dA = \frac{\Delta\phi}{L}GJ$$

$$\Delta\phi = \frac{TL}{GJ},$$

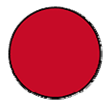
In general $\dfrac{d\phi(z)}{dz} = \dfrac{T(z)}{GJ}$

$$\tau(r) = \frac{G\Delta\phi}{L} r = \frac{T}{J} r$$

## For circular geometry, J is the polar moment of inertia

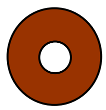
$$J = \int r^2 dA$$

**Circular Shaft:**



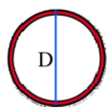
$$J = \frac{\pi D^4}{32}$$

**Hollow Shaft:**



$$J = \frac{\pi}{32}\left(OD^4 - ID^4\right)$$

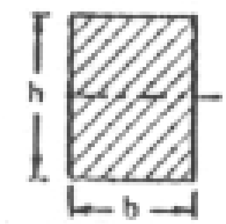
**Thin Walled Tube:**




$$J = \frac{\pi}{4} D^3 t$$

## For non-circular geometry, use K from tables, which is equivalent.

Cross section




$$\Delta\phi = \frac{TL}{GK}$$

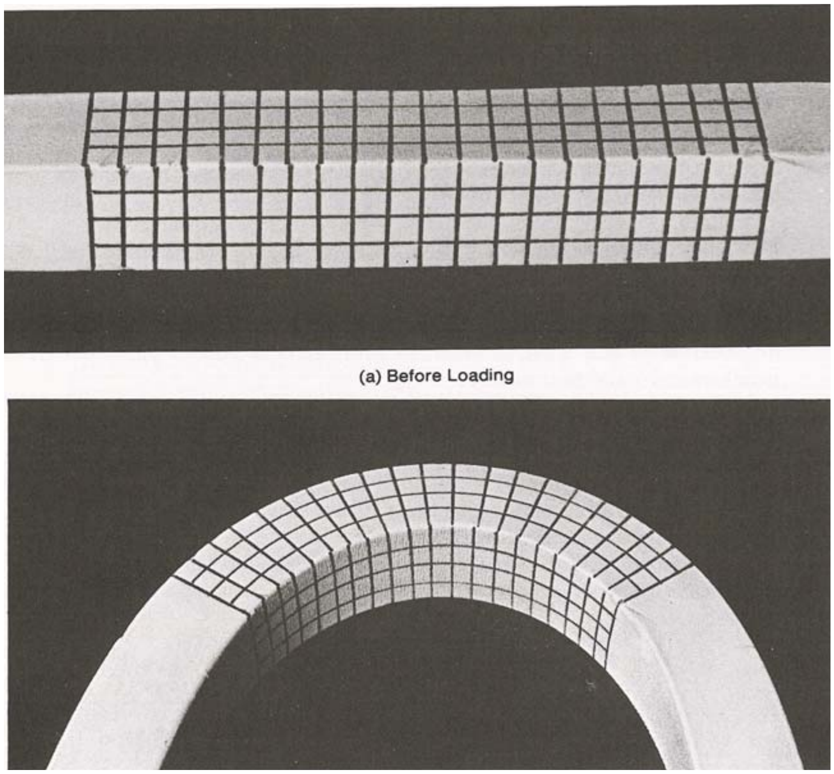
$$K = \frac{1}{3} h b^3 \left(1 - 0.58 \frac{b}{h}\right)$$

$$\tau(r) \cong \frac{T}{K} r$$

For irregular cross sections, torque gets weird.

# Bending

Consider the following rubber beam with grid lines subjected to pure bending, created by a moment.

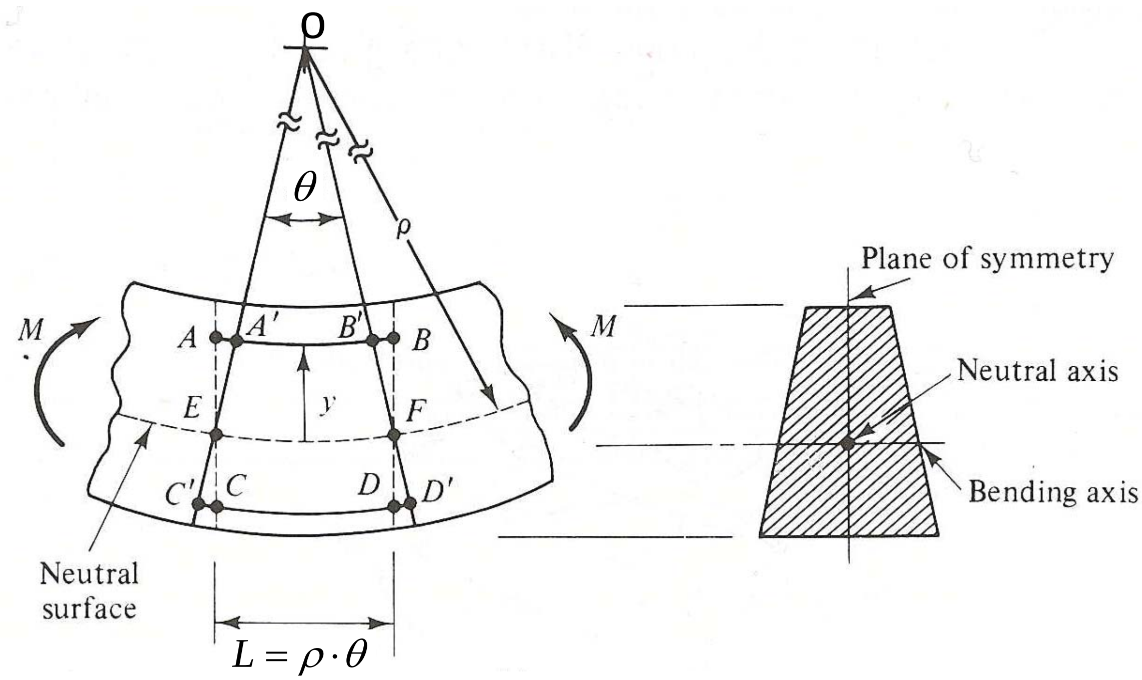


(a) Before Loading

**Where does bending stiffness come from?**

**Look at the material deformation.**

For a bent beam, evaluate over length *L*:




As shown above,

o = center of curvature
ρ = radius of curvature

Using small angle approximation

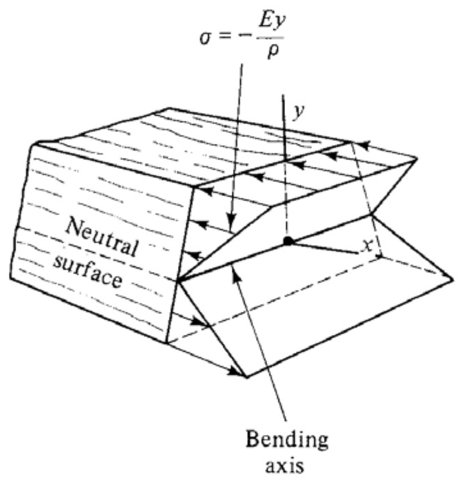
$$\rho = \frac{L}{\theta}$$

**Curvature = 1/radius**

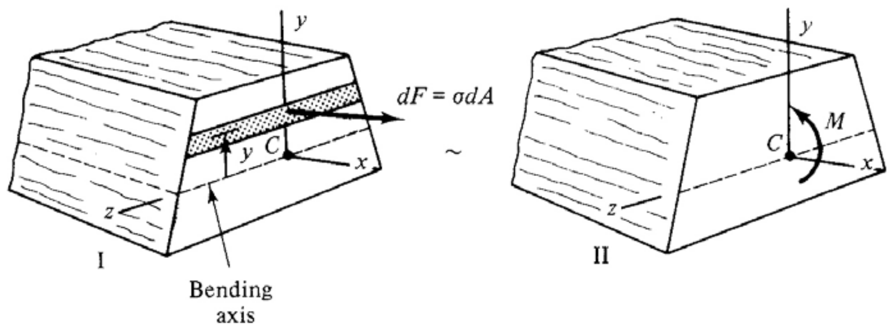
$$\Delta L(y) = \theta \cdot y$$

$$\frac{\Delta L(y)}{L} = \frac{\theta \cdot y}{L}$$

$$= \varepsilon(y)$$



(a)



(b)

Look at a moment acting on a beam of length L,

Each element contributes

$$dM = y \times \sigma(y)dA$$

**From elasticity**

$$\sigma(y) = \varepsilon(y) \cdot E$$

From above,

$$\varepsilon(y) = \frac{\theta \cdot y}{L}$$

$$M = \int dM = \int y\left(\sigma(y)\right)dA$$

$$M = \int dM = \int y\left(\frac{y \cdot \theta}{L} E\right)dA$$

$$M = \frac{\theta}{L} E \int y^2 dA = \frac{\theta}{L} EI$$

$$\theta = \frac{ML}{EI}$$

In general:

$$\frac{d\theta}{dL} = \frac{M}{EI}$$

I is the second moment in the y direction

$$I = \int y^2 dA$$

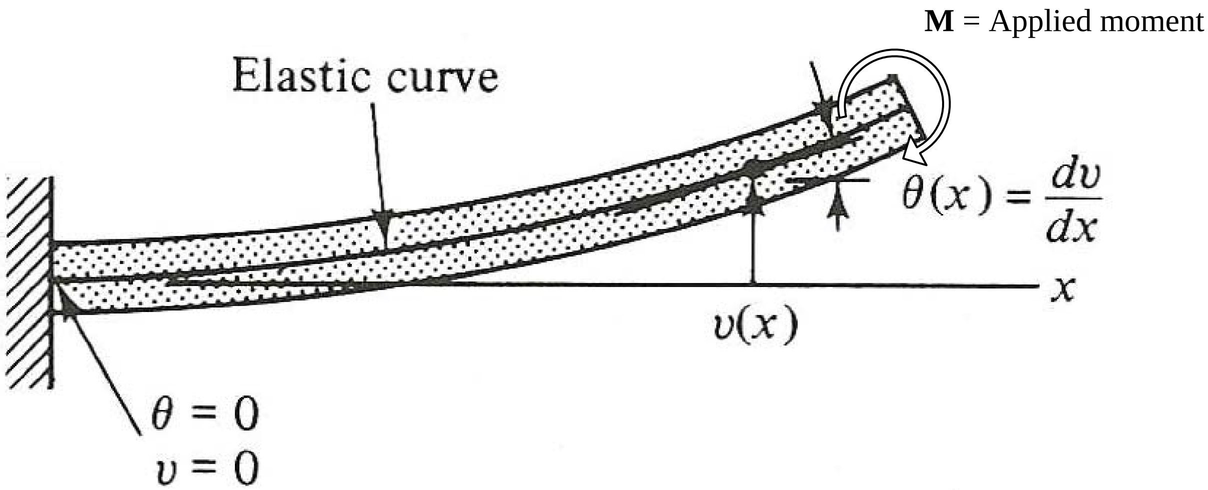
Max stress at max *y*

$$\sigma(y) = \varepsilon(y) \cdot E$$

$$= \theta \frac{y}{L} \cdot E$$

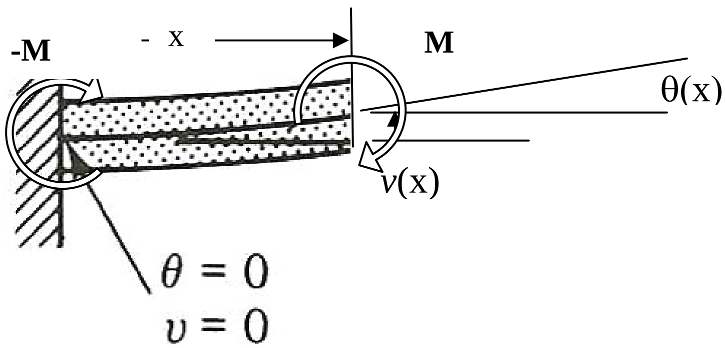
$$= \frac{ML}{EI} \frac{y}{L} E = \frac{My}{I}$$

For a beam in pure bending:




At distance x along beam:
FBD:

$$\frac{d\theta(x)}{dx} = \frac{M(x)}{EI}$$




When M = constant

$$\theta(x) = \frac{dv}{dx} = \frac{Mx}{EI}$$

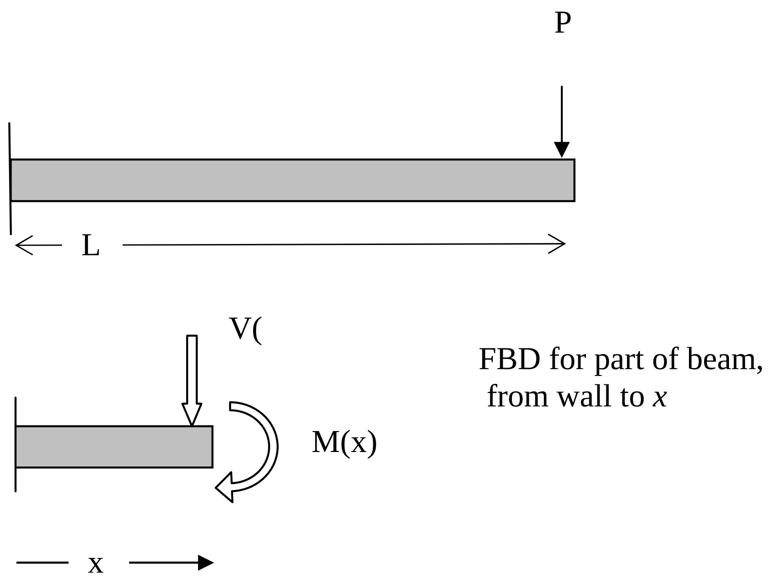
$$v(x) = \int \theta(x) dx$$

$$= \frac{Mx^2}{2EI} + C$$

At x = 0, v = 0 so C=0

Beam bending is always caused by a moment in the beam. This depends on loading and geometry.

In general, the moment varies along the length of the beam, so the net deflection must be integrated.




$$M(x) = F(L-x)$$

$$\Delta\theta = \int \frac{M}{EI}dx = \frac{F}{EI}\int_{x'=0}^{x}(L-x')dx'$$

$$\Delta\theta(x) = \frac{FLx}{EI} - \frac{Fx^2}{2EI}$$

$$\Delta\theta(L) = \frac{FL^2}{2EI}$$

Max stress occurs at max M, which is at x = 0

$$M = FL$$

$$\sigma_{\max} = \frac{My_{\max}}{I} = \frac{FLy_{\max}}{I}$$

To get deflection, integrate angle

$$\frac{d^2y}{dx^2} = \frac{d\theta}{dx} = \frac{M(x)}{EI}$$

$$\Delta y = \int \theta dx$$

For our example:

$$\Delta\theta(x) = \frac{FLx}{EI} - \frac{Fx^2}{2EI}$$

$$y(x) = \int_{x'=0}^{x} \Delta\theta(x')dx'$$

$$= \frac{FLx^2}{2EI} - \frac{Fx^3}{6EI}$$

$$so$$

$$y(L) = \frac{FL^3}{3EI}$$

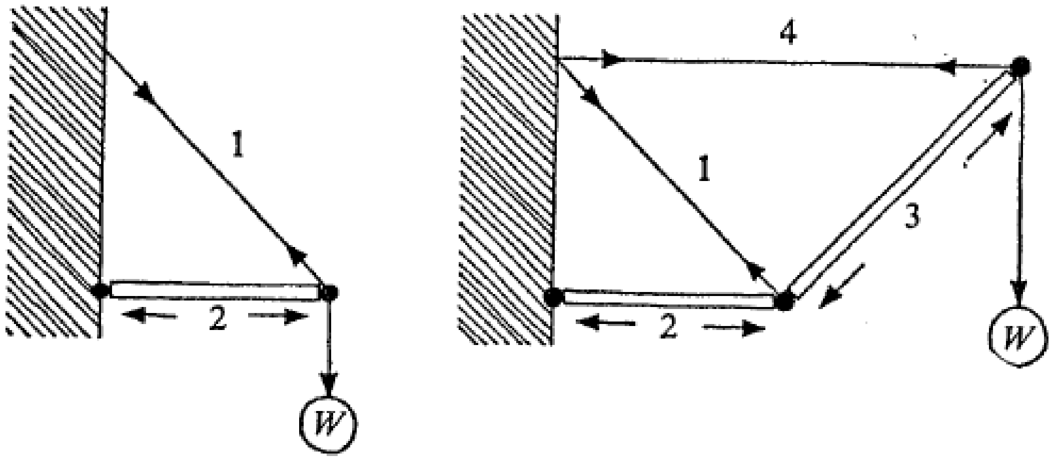
Usually you can use tables to provide these relations

For rectangular cross section

$$I = \frac{1}{12}bh^3$$

Stiffness = dF/dy is proportional $\frac{E \cdot A \cdot h^2}{L^3}$

Understanding beam stresses:



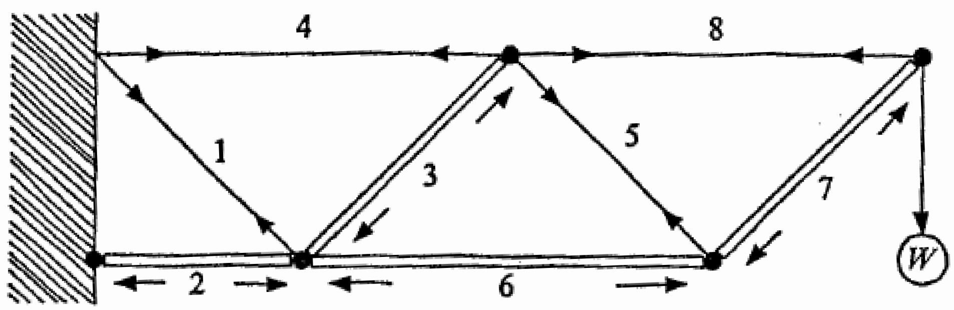
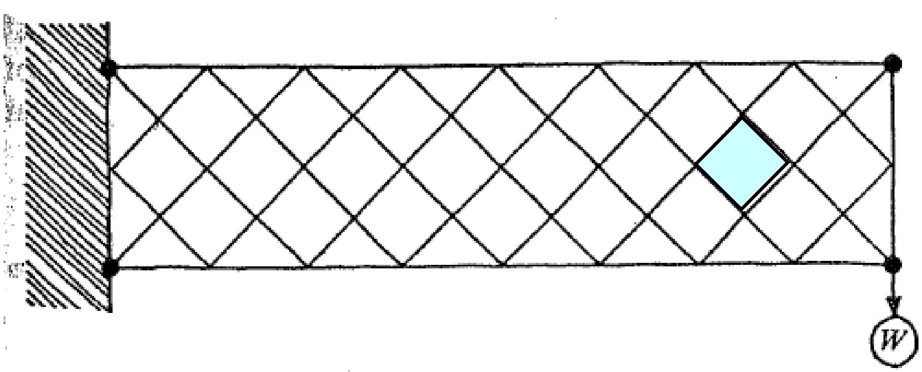




*Figure 21.*



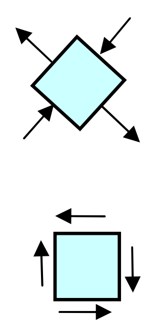
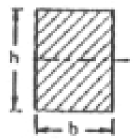
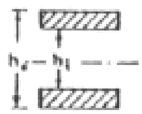
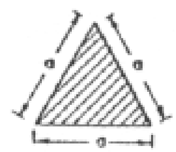
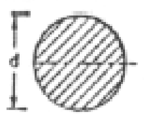
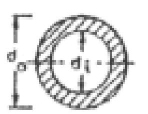
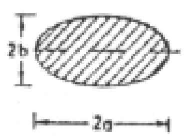
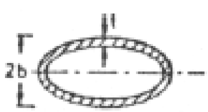
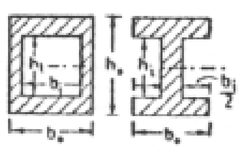
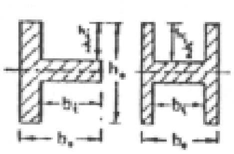
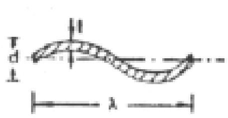
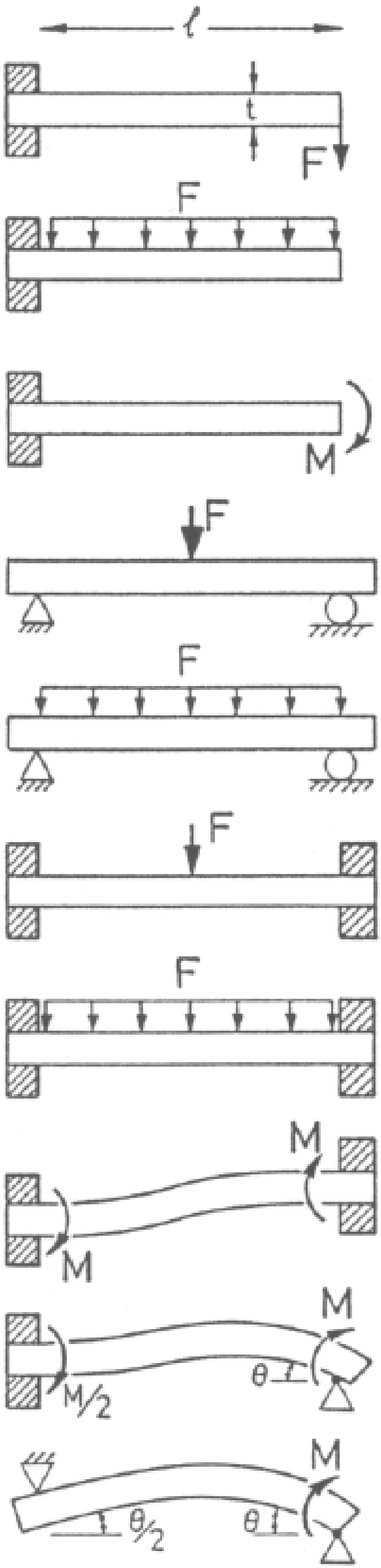



*Figure 22.* The shear can equally well be taken by a multiple lattice or indeed by a continuous plate.

| SECTION | Cross-sectional area<br>A(m²) | Second moment about neutral axis<br>I (m⁴) | Equivalent polar moment<br>K (m⁴) | $I/y_m$ (m³) | H(m³) |
|---|---|---|---|---|---|
|  | $bh$ | $\frac{bh^3}{12}$ | $\frac{1}{3}hb^3\left(1-0.58\frac{b}{h}\right)$<br>(b<h) | $\frac{bh^2}{6}$ | $\frac{bh^2}{4}$ |
|  | $b(h_o - h_i)$ | $\frac{b}{12}(h_o^3 - h_i^3)$ | — | $\frac{b}{12h_o}(h_o^3 - h_i^3)$ | $\frac{b}{4}(h_o^2 - h_i^2)$ |
|  | $\frac{\sqrt{3}}{4}a^2$ | $\frac{a^4}{32\sqrt{3}}$ | $\frac{a^4\sqrt{3}}{80}$ | $\frac{a^3}{32}$ | — |
|  | $\frac{\pi d^2}{4}$ | $\frac{\pi}{64}d^4$ | $\frac{\pi}{32}d^4$ | $\frac{\pi}{32}d^3$ | $\frac{1}{6}d^3$ |
|  | $\frac{\pi}{4}(d_o^2 - d_i^2)$ | $\frac{\pi}{64}(d_o^4 - d_i^4)$ | $\frac{\pi}{32}(d_o^4 - d_i^4)$ | $\frac{\pi}{32d_o}(d_o^4 - d_i^4)$ | $\frac{1}{6}(d_o^3 - d_i^3)$ |
|  | $\pi ab$ | $\frac{\pi}{4}ab^3$ | $\frac{\pi a^3 b^3}{a^2 + b^2}$ | $\frac{\pi}{2}ab^2$ | — |
|  | $2\pi(ab)^{1/2}t$ | $\frac{\pi}{4}ab^3t\left(\frac{1}{a}+\frac{3}{b}\right)$ | $\frac{4\pi t a^2 b^2}{(a+b)}$ | $\frac{\pi a b^2 t}{2}\left(\frac{1}{a}+\frac{3}{b}\right)$ | — |
|  | $h_o b_o - h_i b_i$ | $\frac{1}{12}(b_o h_o^3 - b_i h_i^3)$ | — | $\frac{1}{12h_o}(b_o h_o^3 - b_i h_i^3)$ | $\frac{1}{4}(b_o h_o^2 - b_i h_i^2)$ |
|  | $h_o b_i + h_i b_o$ | $\frac{1}{12}(b_i h_o^3 + b_o h_i^3)$ | — | $\frac{1}{12h_o}(b_i h_o^3 + b_o h_i^3)$ | $\frac{1}{4}(b_o h_o^2 + b_i h_i^2 - 2bh_i h_o)$ |
|  | $t\lambda\left(1+\left(\frac{\pi d}{2\lambda}\right)^2\right)$ | $\frac{t\lambda d^2}{8}\left(1-\frac{0.81}{1+2.5\left(\frac{d}{2\lambda}\right)^2}\right)$ | — | — | — |




| $C_1$ | $C_2$ |
|---|---|
| 3 | 2 |
| 8 | 6 |
| 2 | 1 |
| 48 | 16 |
| $\frac{384}{5}$ | 24 |
| 192 | – |
| 384 | – |
| 6 | – |
| – | 4 |
| – | 3 |

$$\delta = \frac{F\ell^3}{C_1 E I} = \frac{M\ell^2}{C_1 E I}$$

$$\theta = \frac{F\ell^2}{C_2 E I} = \frac{M\ell}{C_2 E I}$$

E = YOUNGS MODULUS ($N/m^2$)

δ = DEFLECTION (m)

F = FORCE (N)

M = MOMENT (Nm)

ℓ = LENGTH (m)

b = WIDTH (m)

t = DEPTH (m)

θ = END SLOPE (–)

I = SEE TABLE 2 ($m^4$)

y = DISTANCE FROM N.A.(m)

R = RADIUS OF CURVATURE (m)

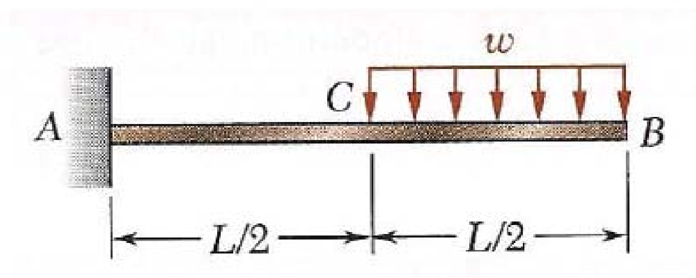
$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

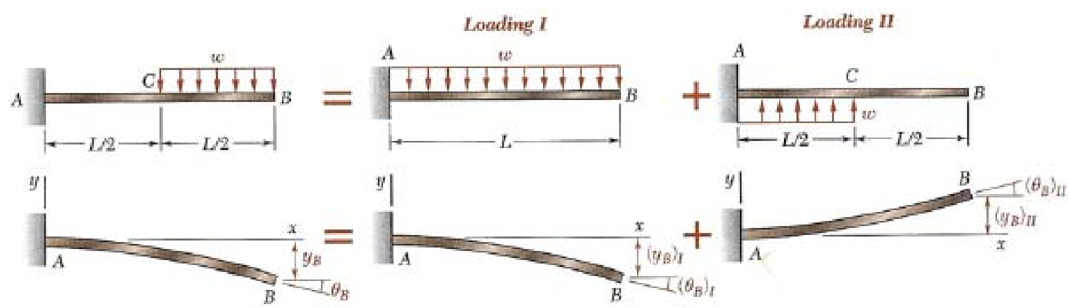
## Linearity and superposition

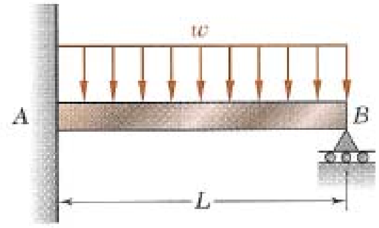
For small deflections, you can find the net effect from multiple forces by

1. Compute deflections from each force, ignoring the other forces
2. Add the deflections

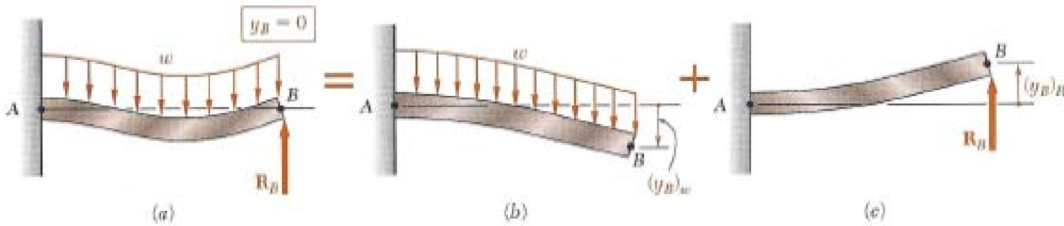
This is true for both linear and angular deflections






Treating the reaction at B as the redundant support, we have:




Determine:

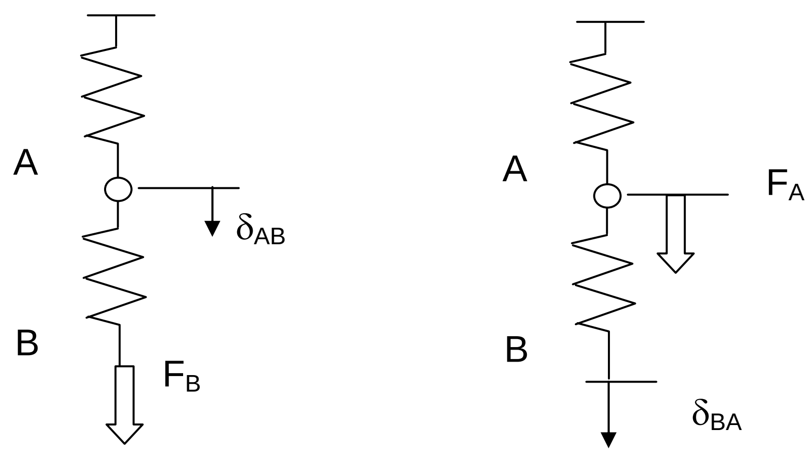
1. The reaction force at location "B".
2. The moment at location "A".

## Reciprocity

### For two points *A* and *B*

Consider two reciprocal cases:

A force $F_A$ applied at A gives deflection at B $\delta_{BA}$
A force $F_B$ applied at B gives deflection at A $\delta_{AB}$

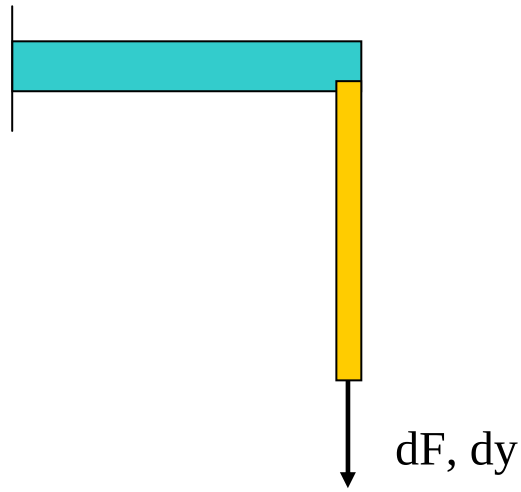



It is generally true that

$$\delta_{BA} / F_A = \delta_{AB} / F_B$$

## Resonant frequency

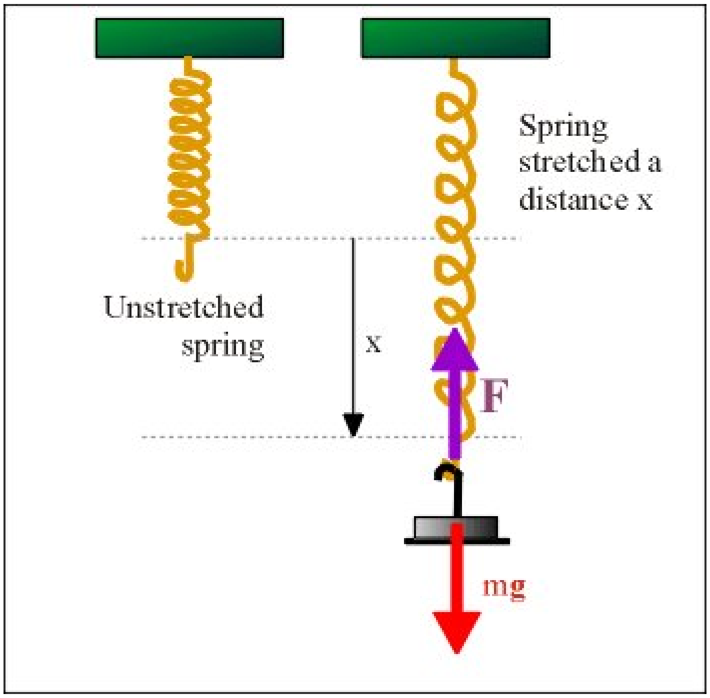
You can always find the stiffness as the ratio of $\dfrac{\Delta F}{\Delta y}$




Resonant frequency for a spring-mass system is

$$\omega_n = \sqrt{\frac{k}{m}}$$ (units are radians/sec)

To find the resonant frequency, you need the ratio of the effective stiffness to the effective mass.




The force applied is due to the mass and gravity

$$F = m \cdot g$$

The deflection of the spring depends on stiffness *k*

$$F = k \cdot \delta x$$

So

$$m \cdot g = k \cdot \delta x$$

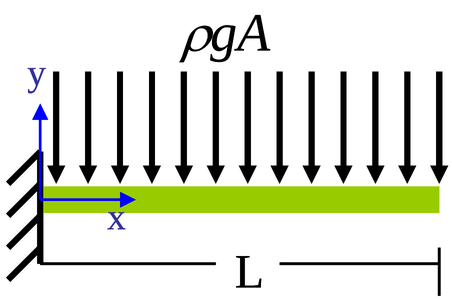
$$\sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta x}} = \omega_n$$

For 1 mm deflection, $\omega_n$ = 100 rad/s = 16 Hz.

# Specific stiffness

Self-weight deflection – resulting solely from gravity acting on the structure's mass

- Self weight deflection is proportional density ρ and inversely proportional to the elastic modulus E.
- Specific Stiffness (E/ρ) is the ratio of the modules of elasticity to density -- Higher is better




$$F = \rho g A L$$

$$\delta_y = \frac{FL^3}{8EI} = \frac{\rho g A L^4}{8EI}$$

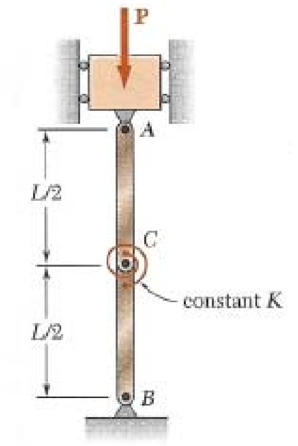
Resonant frequency also depends on specific stiffness

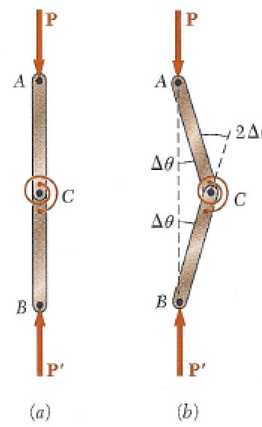
$$\omega_n = 3.516\sqrt{\left(\frac{EI}{\rho A L^4}\right)}$$

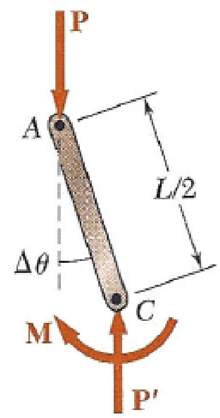
The modal stiffness is always proportional to E and the modal mass is always proportional to ρ.

# Buckling

Compressive forces, coupled with deflections can create unstable members



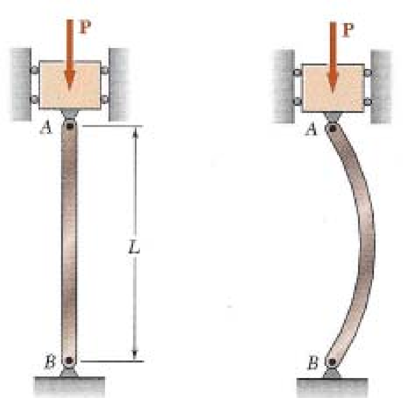




Restoring moment from spring = K (2Δθ)

Moment from axial force and geometry = PL/2 Δθ

Unstable for $P_{CRIT} > 4K/L$

Column stability, same idea




$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

## Other ends:

replace $L$ with $L_e$