Introduction to Solid Mechanics Concepts and Fundamental Relationships

OPTI 521: Homework 7, Part I
23 October, 2006

Abstract

A basic summary of solid mechanics is provided. The mechanics of stress and strain is introduced. These basic concepts are used to explain the physical meaning of Poisson's ratio, bulk modulus, yield strength, and precise elastic limit.

Stress and Strain

Stress is defined as the force per unit area. Normal stress is stress applied normal to a surface while shear stress is applied in the tangent direction to a surface, see figures 1 and 2. These principles can also be applied to infinitesimally small volumes internal to a material. Normal strain is the ration of the change in length to the original length. Shear strain is the angle that the material deflects in shear strain, γ, see figure 2. Strain is unit-less because it is an angle, in radians, or it is a ratio of lengths. Stress has the units of force/length^2, typically lb/in^2 (psi) or N/m^2. N/m^2 is equivalent to the SI unit of Pascal (Pa).

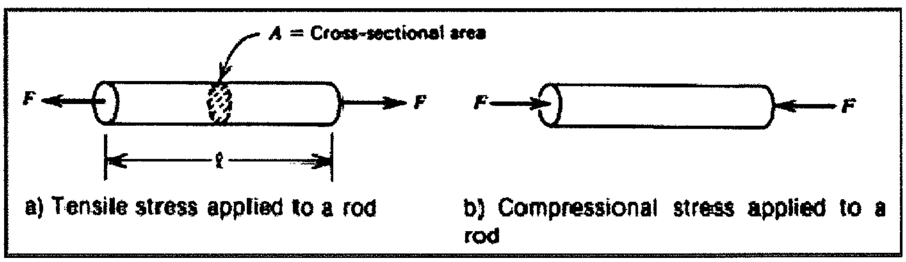

Figure 1. Normal stress applied to a beam

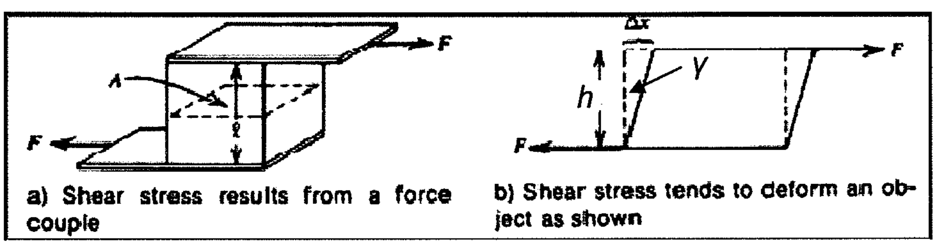

Figure 2. Shear stress applied to a solid

Normal Stress: $\sigma = \dfrac{F}{A}$ Shear Stress: $\tau = \dfrac{F}{A}$

Normal Strain: $\varepsilon = \dfrac{\Delta L}{L}$ Shear Strain: $\gamma = \dfrac{\Delta x}{h}$

Normal stress is related to normal strain with a simple formula, shear stress and strain are related with a very similar formula. The proportionality constant, E, is called the modulus of elasticity, or Young's modulus. Similarly, G is called the shear modulus. Young's modulus is a fundamental property of a material, its units are the same as stress.

Normal Stress-Strain: $\sigma = \varepsilon E$ Shear Stress-Strain: $\tau = \gamma G$

Shear Modulus: $G = \dfrac{E}{2(1+\nu)}$; ν is defined in the next section

Compressibility

When normal stress is applied to a solid it deforms according to the formulas above. When rod is pulled, for example, it gets longer, see figure 3. As the rod gets longer it also gets skinnier. Poisson's ratio is the fundamental material property that is the ratio of these changes. The theoretical upper limit for Poisson's ratio is 0.5; this corresponds to an incompressible solid. Typically, Poisson's ratio is around 0.3 for metals.

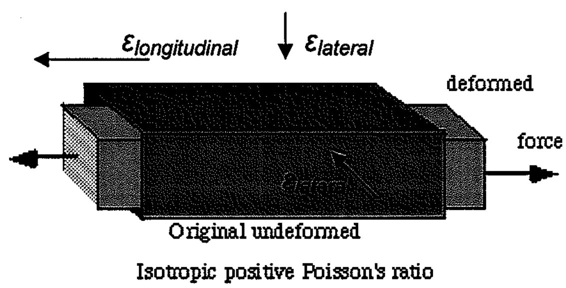

Figure 3: Deformation in 3 dimensions showing all 3 strains

Poisson's Ration: $\nu = -\dfrac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}}$

In addition to deforming in all directions per Poisson's ratio, when a solid material is stressed its volume will change (assuming its Poisson's ratio is not 0.5). When equal positive pressure is applied to all sides of the material, the strain is negative, or inward, in all directions. From this principle the bulk modulus is calculated. The bulk modulus is the compressibility of a material. A large bulk modulus is relatively incompressible.

Bulk Modulus: $k = \dfrac{E}{3(1-2\upsilon)}$

Note that as the Poisson's ratio approaches 0.5 the bulk modulus approaches infinity.

Yield Strength

Many materials, including metals, start to yield before they break. There are two principle characteristics of yielding. First, the material begins to permanently deform, as opposed to elastically deform. Second, the stress-strain relationship is no longer linear. Figure 4 shows a typical stress-strain plot that would be collected by using an INSTRON machine, for example. This figure is used to define the yield strength and the precision elastic limit for a material.

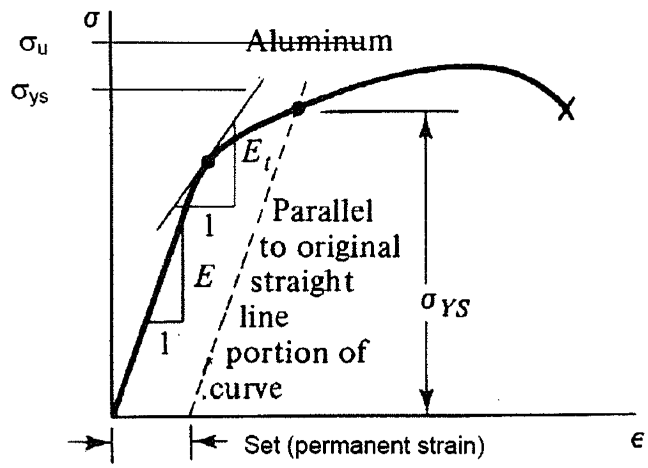

Figure 4: Typical stress-strain diagram (Aluminum shown)

The engineering yield stress is established with the .2% offset method. The yield stress is arbitrarily chosen to be the stress that has a permanent deformation strain of .002, see figure 4. The precision elastic limit is defined as the stress that has a permanent deformation strain of 1 ppm.

Sources

National Instruments: http://zone.ni.com/devzone/cda/ph/p/id/250

Rod Lakes, University of Wisconsin:
http://silver.neep.wisc.edu/~lakes/PoissonIntro.html

HW 6 - PART 3 - BEAM DEFLECTION - SOLUTION

[A] CALCULATE CROSS SECTIONAL AREA A, SECOND MOMENTS I_x, I_y, POLAR MOMENT OF INERTIA K

$$A = w \cdot h = \boxed{0.5\,\text{IN}^2}$$

$$I_x = \frac{wh^3}{12} = \frac{(.5)(1\,\text{IN})^3}{12} = \boxed{0.04166\,\text{IN}^4}$$

$$I_y = \frac{hw^3}{12} = \frac{(1\,\text{IN})(.5\,\text{IN})^3}{12} = \boxed{0.01041\,\text{IN}^4}$$

$$K = \frac{1}{3}hb^3\left(1 - 0.58\frac{b}{h}\right) = \frac{1}{3}(1\,\text{IN})(.5\,\text{IN})^3\left(1 - .58\left(\frac{.5\,\text{IN}}{1\,\text{IN}}\right)\right) = \boxed{0.02958\,\text{IN}^4}$$

[B] FOR EACH CASE FIND MAX θ, MAX δ, MAX σ.

(A) $$\Delta L = \frac{FL}{EA} = \frac{(10\,\text{lb})(10\,\text{IN})}{(10\times10^6\,\text{PSI})(0.5\,\text{IN}^2)} = \boxed{2\times10^{-5}\,\text{IN}}$$

$$\sigma = \frac{F}{A} = \frac{10}{.5\,\text{IN}^2} = \boxed{20\,\text{PSI}} \qquad \boxed{\theta = 0}$$

(B) $$\delta = \frac{FL^3}{3EI_x} = \frac{10(10)^3}{3(10\times10^6)(.04166)} = \boxed{8\times10^{-3}\,\text{IN}}$$

$$\theta = \frac{FL^2}{2EI_x} = \frac{10(10)^2}{2(10\times10^6)(.04166)} = \boxed{1.2\times10^{-3}\,\text{IN}}$$

$$\sigma = \frac{M\,y_{MAX}}{I_x} = \frac{(10\cdot10)(0.5)}{.04166} = \boxed{1200\,\text{PSI}}$$

(C) $$\delta = \frac{FL^3}{3EI_y} = \frac{10(10)^3}{3(10.3\times10^6)(.01041)} = \boxed{3.1\times10^{-2}\,\text{IN}}$$

$$\theta = \frac{FL^2}{2EI_y} = \frac{10(10)^2}{3(10.3\times10^6)(.01041)} = \boxed{4.7\times10^{-3}\,\text{RAD}}$$

$$\sigma = \frac{M\,y_{MAX}}{I_y} = \frac{(10\cdot10)(0.25\,\text{IN})}{.01041} = \boxed{2400\,\text{PSI}}$$

(D) $$\delta = \frac{ML^2}{2EI_x} = \frac{(10)(10_{IN})^2}{2(10.3\times10^6 PSI)(.04166)} = \boxed{1.2\times10^{-3}\,IN}$$

$$\theta = \frac{ML}{EI_x} = \frac{(10_{IN\cdot lb})(10_{IN})}{(10.3\times10^6 PSI)(.04166)} = \boxed{2.3\times10^{-4}\,RAD}$$

$$\sigma = \frac{M\,Y_{MAX}}{I_x} = \frac{(10_{IN\cdot lb})(0.5)}{.04166} = \boxed{120\ PSI}$$

(E) $$\delta = \frac{ML^2}{2EI_y} = \frac{(10_{IN\cdot lb})(10_{IN})^2}{2(10.3\times10^6 PSI)(\cancel{.04166}\ .01041)} = \boxed{4.7\times10^{-3}\,IN}$$

$$\theta = \frac{ML}{EI_y} = \frac{(10_{IN\cdot lb})(10_{IN})}{(10.3\times10^6 PSI)(.01041)} = \boxed{9.3\times10^{-4}\,RAD}$$

$$\sigma = \frac{M\,Y_{MAX}}{I_y} = \frac{(10_{IN\cdot lb})(.25)}{.01041} = \boxed{240\ PSI}$$

(F) $$G = \frac{E}{2(1+\nu)} = \frac{10.3\times10^6\ PSI}{2(1+0.23)} = 4.19\times10^6\ PSI$$

$$\Delta\phi = \frac{ML}{GK} = \frac{(10_{IN\cdot lb})(10_{IN})}{(4.19\times10^6 PSI)(.029583_{IN^4})} = \boxed{8.1\times10^{-4}\,RAD}$$

$$\tau = \frac{Tr}{2K} = \frac{(10_{IN\cdot lb})\left(\sqrt{1_{IN}^2 + 0.5_{IN}^2}\right)}{2(.029583_{IN^4})} = \boxed{189\ PSI}$$

(G) $$\delta = \frac{FL^3}{8EI_x} = \frac{(10\,lb)(10)^3}{8(10.3\times10^6 PSI)(.04166_{IN^4})} = \boxed{2.9\times10^{-3}\,IN}$$

$$\theta = \frac{FL^2}{6EI_x} = \frac{(10\,lb)(10_{IN})^2}{6(10.3\times10^6 PSI)(.04166_{IN^4})} = \boxed{3.9\times10^{-4}\,RAD}$$

$$\sigma = \frac{M\,Y_{MAX}}{I_x} = \frac{(10\,lb\cdot 5_{IN})(0.5_{IN})}{(0.04166_{IN^4})} = \boxed{600\ PSI}$$

(H)

$$\delta = \frac{Fx^2}{6EI_x}(3L - x)\Bigg|_{x=\frac{L}{2}} = \frac{10\,lb(5\,in)^2}{6(10.3\times10^6\,psi)(.04166\,in^4)}\left(3(10\,in) - 5\,in\right) = \boxed{2.4\times10^{-3}\,in}$$

$$\theta = \frac{Fx^2}{2EI_x}\Bigg|_{x=\frac{L}{2}} = \frac{(10\,lb)(5\,in)^2}{2(10.3\times10^6\,psi)(.04166\,in^4)} = \boxed{2.9\times10^{-4}\,rad}$$

$$\sigma = \frac{My_{max}}{I_x} = \frac{(10\,lb\cdot 5\,in)(0.5\,in)}{.04166\,in^4} = \boxed{600\,psi}$$

(I)

$$\delta = \frac{FL^3}{48EI_x} = \frac{(10\,lb)(10\,in)^3}{48(10.3\times10^6\,psi)(.04166\,in^4)} = \boxed{4.8\times10^{-4}\,in}$$

$$\theta = \frac{FL^2}{16EI_x} = \frac{(10\,lb)(10\,in)^2}{16(10.3\times10^6\,psi)(.04166\,in^4)} = \boxed{1.5\times10^{-4}\,rad}$$

$$\sigma = \frac{My_{max}}{I_x} = \frac{\left(\frac{FL}{4}\right)(.5)}{I_x} = \frac{\left(\frac{10\,lb\cdot 10\,in}{4}\right)(0.5\,in)}{.04166\,in^4} = \boxed{300\,psi}$$

C FIND RESONANT FREQUENCY FOR 3 MODES, $k = \frac{F}{\delta}$

$$f = \frac{1}{2\pi}\sqrt{\frac{k\cdot 384\,\frac{lb_m\cdot in}{g^2}}{10\,lb_m}} = \frac{1}{2\pi}\sqrt{\frac{F/\delta\cdot 384\,lb_m\cdot in/g^2}{10\,lb_m}} = \frac{1}{2\pi}\sqrt{\frac{384\,\frac{in}{s^2}}{\delta}}$$

X-MODE

$$f_x = \frac{1}{2\pi}\sqrt{\frac{384\,\frac{in}{s^2}}{8\times10^{-3}\,in}} = \boxed{35\,Hz}$$

Y-MODE

$$f_y = \frac{1}{2\pi}\sqrt{\frac{384\,in/s^2}{3.1\times10^{-2}\,in}} = \boxed{18\,Hz}$$

AXIAL MODE

$$f_A = \frac{1}{2\pi}\sqrt{\frac{384\,in/s^2}{2\times10^{-5}\,in}} = \boxed{700\,Hz}$$