TRACK I

WRITTEN PRELIM EXAM – FIRST DAY

Fall 2004

September 21, 2004
8:30 a.m. to 12:00 p.m.

Answer all six questions. Start each answer on a new page.

In the upper right hand corner of each sheet you hand in, put your code number (NO NAMES PLEASE) and the problem number. Staple together all sheets for a given problem.

Insert your answers into the manila envelope supplied and note which problems you have answered on the outside of the envelope.

The following are some helpful items:

$h = 6.625 \times 10^{-34}\ \mathrm{J \cdot s} = 4.134 \times 10^{-15}\ \mathrm{eV \cdot s}$

$e = 1.6 \times 10^{-19}\ \mathrm{C}$

$c = 3.0 \times 10^{8}\ \mathrm{m/s}$

$k_B = 1.38 \times 10^{-23}\ \mathrm{J/K}$

$\sigma = 5.67 \times 10^{-8}\ \mathrm{W/K^4 \cdot m^2}$

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}$

$\mu_0 = 1.26 \times 10^{-6}\ \mathrm{H/m}$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$2\cos A \cos B = \cos(A - B) + \cos(A + B)$

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$2\sin A \cos B = \sin(A + B) + \sin(A - B)$

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$\sin 2A = 2 \sin A \cos A$

$\cos 2A = 2\cos^2 A - 1$

$\cos 2A = 1 - 2\sin^2 A$

$\sin^2(\frac{A}{2}) = \frac{1}{2}(1 - \cos A)$

$\cos^2(\frac{A}{2}) = \frac{1}{2}(1 + \cos A)$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$\nabla\phi\psi = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\phi$

$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

$\nabla \cdot (\nabla \times \mathbf{F}) = 0$

$\nabla \times (\phi\mathbf{F}) = \phi(\nabla \times \mathbf{F}) + \nabla\phi \times \mathbf{F}$

$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

$\nabla \times \nabla\phi = 0$

$\oint_S (\mathbf{F} \cdot \mathbf{n})\, da = \int_V (\nabla \cdot \mathbf{F})\, d^3x$

$\oint_C \mathbf{F} \cdot d\ell = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da$

$\oint_S \phi\mathbf{n}\, da = \int_V \nabla\phi\, d^3x$

$\oint_S \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_V [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

$\oint_S (\mathbf{n} \times \mathbf{F})\, da = \int_V (\nabla \times \mathbf{F})\, d^3x$

Consider a linear, isotropic, and homogeneous medium for which the electric displacement $\vec{D}(t)$ can be expressed in terms of the time history of the electric field $\vec{E}(t)$ as

$$\vec{D}(t) = \epsilon_0 \vec{E}(t) + \int_{-\infty}^{\infty} d\tau \mathcal{R}(\tau) \vec{E}(t - \tau), \tag{1}$$

with $\mathcal{R}(\tau)$ the linear response function, and the spatial dependence of the fields has been suppressed for simplicity in notation.

(a) What condition must be imposed on the linear response function $\mathcal{R}(\tau)$ so that optical response described by Eq. (1) is causal? (1 points)

(b) By considering a monochromatic electric field of frequency ω show that Eq. (1) leads to the frequency-dependent electric displacement $\vec{D}(\omega) = \epsilon(\omega)\vec{E}(\omega)$, and derive an expression for the frequency-dependent dielectric constant $\epsilon(\omega)$ in terms of the linear response function. (3 points)

(c) Using the results from part (b) calculate the frequency dependence of the dielectric constant $\epsilon(\omega)$ for an instantaneously responding medium with $\mathcal{R}(\tau) = C\delta(\tau)$, C being a constant. (2 points)

(d) Next consider the linear response function $\mathcal{R}(\tau) = C\gamma\theta(\tau)\exp(-\gamma\tau)$, with $\gamma > 0$ and C constants, and $\theta(\tau)$ the Heaviside function which is zero for negative arguments and unity for positive arguments. Using the results from part (b) show that the corresponding dielectric constant is $\epsilon(\omega) = \epsilon_0 + C\gamma/(\gamma - i\omega)$. (3 points)

(e) In what frequency limit can the frequency-dependent dielectric constant from part (d) be approximated as arising from an instantaneously responding medium as in part (c)? (1 points)

The Telegrapher's equation for propagation of an electric field $\vec{E}(\vec{r},t)$ in a conducting medium of background refractive-index n occupying the half-space $z > 0$ is

$$\nabla^2\vec{E} - \nabla(\nabla\cdot\vec{E}) - \frac{n^2}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \mu_0\sigma\frac{\partial\vec{E}}{\partial t} = 0,$$

where $\sigma > 0$ is the conductivity.

(a) By considering a monochromatic plane-wave electric field

$$\vec{E}(\vec{r},t) = \frac{\hat{e}}{2}\left[E_t e^{i(Kz-\omega t)} + c.c.\right], \quad z > 0$$

show that the complex propagation wavevector $K = k_r + i\alpha/2$, with k_r and α real, obeys $K^2 = n^2\omega^2/c^2 + i\omega\mu_0\sigma$, stating any assumptions made. (3 points)

(b) Next assume that a field is incident from vacuum onto the conducting medium from the half-space $z < 0$ and that some is also reflected

$$\vec{E}(\vec{r},t) = \frac{\hat{e}}{2}\left[(E_i e^{i\omega z/c} + E_r e^{-i\omega z/c})e^{-i\omega t} + c.c.\right], \quad z < 0.$$

By demanding that the field and its first z-derivative be continuous at the interface at $z = 0$ derive an expression for the field reflectivity coefficient $r = E_r/E_i$. (4 points)

(c) Using the results from part (b), find approximate expressions for the field reflectivity coefficient r in the two limits $\sigma \to 0$ and $\sigma \to \infty$. (3 points)

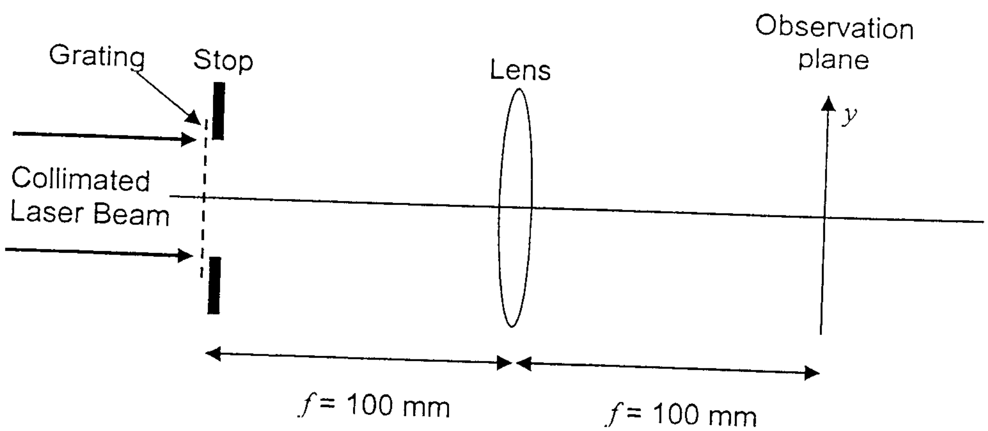
Fall 2004

A camera objective is telecentric in object space and designed to image an object of size ±10 mm onto a detector of size ±5 mm. The working distance (from the object to the objective) is 125 mm. The system must be unvignetted over this field of view and have an image-space working F-number of f/4.

Model the objective as a single thin lens in air.

Determine the system layout including the position and diameter of the thin lens and the position and diameter of the system stop.

The optical system below is used to analyze a laser beam. The laser passes through a rectangular stop with dimension w = 10 mm in the y direction. A grating is placed before the stop with period Λ = 10 μm. The grating produces diffracted orders in the y direction. You may simplify the analysis by only calculating values in the y dimension.



1.) (2 pts) Assume that the grating is removed. If λ = 0.5 μm and f = 100 mm, what is the width focused spot irradiance in the observation plane? (Calculate the width of the central bright lobe, where width is defined as the distance between points of zero irradiance.)

2.) (Assume that the grating is placed as shown in the figure.)

a.) (1 pt) What is the separation between the m = 0 order and the m = 1 order in the observation plane?

b.) (1 pt) What is the separation between the m = 0 order and the m = 2 order in the observation plane?

3.) (2 pts) What is the smallest wavelength change that can be measured if the minimum detectable position change of the m = 1 order in the observation plane is one-half the spot diameter? This is the resolution $\Delta\lambda_{RES}$ of the system.

4.) (2 pts) If a second wavelength is added to the laser beam, how far can the wavelength be increased before the m = 1 order of the second wavelength overlaps the m = 2 order of the λ = 0.5 μm component?

5.) (1 pt) The difference between the wavelength found in (4) and λ = 0.5 μm is the free spectral range $\Delta\lambda_{FSR}$. What is the free spectral range for this grating?

6.) (1 pt.) What is the Finesse of the system, where Finesse = $\dfrac{\Delta\lambda_{FSR}}{\Delta\lambda_{RES}}$?

Prelim Question

(20 Points)
(a) Write down the dielectric function for longitudinal waves.

(20 Points)
(b) Write down the dielectric function for transverse waves.

(20 Points)
(c) Write down the expression for dielectric function as a function of frequency in the optical frequency regime in the oscillator model.

(20 Points)
(d) Use (c) to show that metals are a good reflector of visible light.

(20 Points)
(e) How does metals respond to UV light? Why?

The object shown below left, that is a collection of nine small light sources (essentially point sources), is being imaged by a Newtonian telescope system (primary mirror with flat secondary) onto a CCD-array detector. The center light source is along the optical axis of the imaging system.

(1 point)

A) List the five third order Seidel aberrations. Of these five, give one that can be fully corrected in this telescope and how you would correct for this aberration?

(5 points)

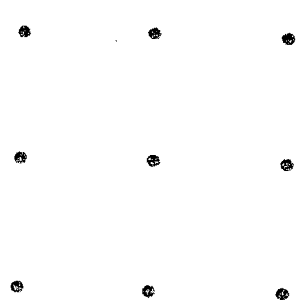
B) The transverse ray fans (ϵ_y versus y_p and ϵ_x versus x_p) for the system for H=1 are shown below. Based on these ray fans, sketch (do not be concerned with quantitative diagrams) the image that you might expect from this system assuming that the detector array is not the limiting factor in the image quality. Explain why your image appears the way it does.

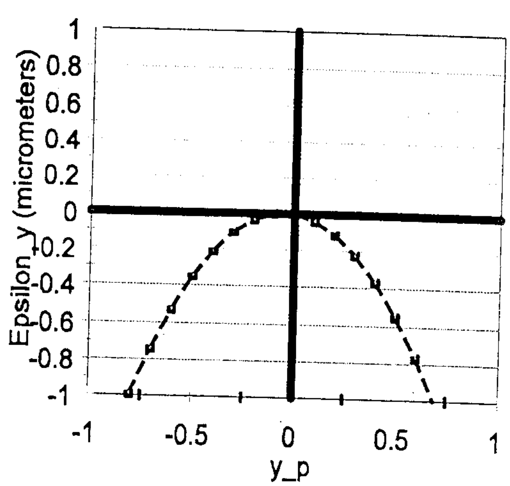
(2 points)

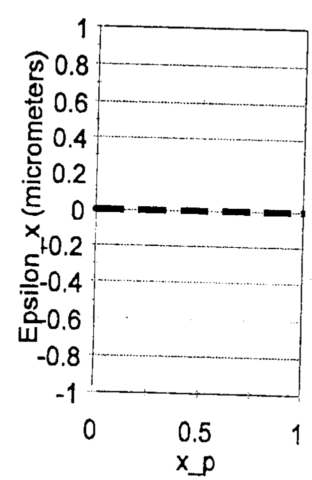
C)The telescope designer opts to reduce the size of the secondary mirror in the telescope to improve the throughput. Describe what effects, if any, this change will have on the image shape.

(2 points)

D) The telescope designer opts to decrease the f/# of the system by a factor of 2. Why would she do this and what quantitative impact will this have on the image quality?







TRACK I

WRITTEN PRELIM EXAM – SECOND DAY

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$\oint_S (\mathbf{n} \times \mathbf{F})\, da = \int_V (\nabla \times \mathbf{F})\, d^3x$

Statistics and stochastic processes
Part I:

Consider two continuous, independent random variables (r.v.'s) x and y characterized by the probability density functions, $p_X(x)$ and $p_Y(y)$, respectively. The r.v. x has mean μ_x and variance σ_x^2, and the r.v. y has mean μ_y and variance σ_y^2. Now, consider the r.v. $z = \frac{1}{2}x - y$.

1. (10%) What is the mean and the variance of the r.v. z?
2. (10%) What is the covariance between x and z?
3. (20%) What is the characteristic function of the r.v. z in terms of $\phi_X(\omega)$ and $\phi_Y(\omega)$, the characteristic functions for the r.v.'s x and y, respectively?
4. (20%) Using the information you derived above, what is $\phi'_Z(0)$ and $\phi''_Z(0)$ (the first and second derivatives of $\phi_Z(\omega)$ evaluated at 0)?

Part II:

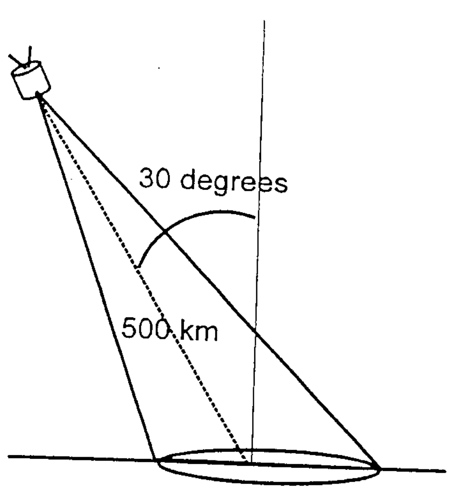
Consider the stochastic process $f(t; a, \phi) = a\cos(t + \phi)$, where a and ϕ are independent random variables. The r.v. a is Poisson distributed with mean μ_a, and the r.v. ϕ is uniformly distributed between $-\pi$ and π.

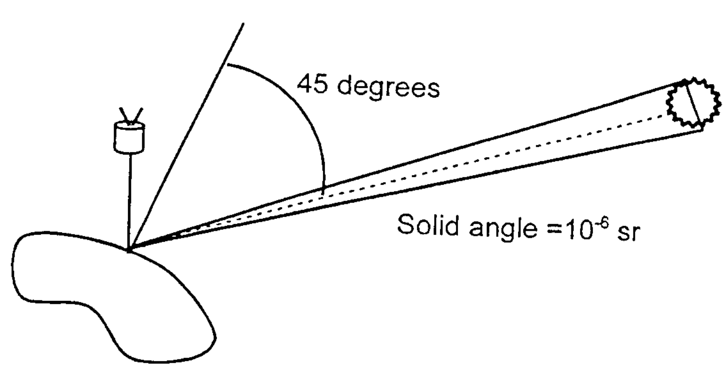
5. (20%) What is the mean of the stochastic process $f(t; a, \phi)$?
6. (20%) What is the autocorrelation of $f(t; a, \phi)$? (Useful trig identity: $\cos(A)\cos(B) = \frac{1}{2}(\cos(A - B) + \cos(A + B))$) Is this stochastic process wide-sense stationary?

A satellite sensor is passing by an asteroid at an altitude of 500 km above the surface and reports an output of 1.00 volts when viewing a 2.888 km^2 area of the asteroid from an angle of 30 degrees from the normal to the asteroid's surface. The area, as shown in the figure below left, is the physical area of the surface seen at the 30-degree view angle and the 500 km distance between the center of the area and the satellite. The only significant light source on the asteroid is a distant star that illuminates the asteroid at an angle of 45 degrees and the star subtends a solid angle of 10^{-6} sr.

The satellite sensor is also pointed at the star for calibration purposes shortly after viewing the asteroid. A neutral density filter with a transmittance of 10^{-6} is placed in front of the sensor's detector and in this configuration the sensor reports an output of 4.44 volts when viewing the star.

Compute the reflectance of the asteroid if there is no atmosphere.





Consider the energy levels of a hydrogen atom defined by $|n,l,m\rangle$, where n is the principle quantum number, l is the electron orbital angular momentum quantum number, and m is the quantum number for the projection of electron orbital angular momentum along a quantization axis, which we'll say is the z axis. For now, ignore other physical quantities that split the $|n,l,m\rangle$ states.

(a) List the possible values of l for n=3.

(b) List the possible values of m for l=2.

(c) Into what $|n,l,m\rangle$ state will the state $|c\rangle = |3,2,2\rangle$ decay by emission of optical radiation? Specify this state by indicating its values of n,l,m. For the rest of this problem, this state will be referred to as state $|b\rangle$.

(d) What is the optical wavelength associated with the transition from $|c\rangle$ to $|b\rangle$? (HINT: The wavelength associated with the ionization energy of H is 87 nm.)

(e) In the transition from $|c\rangle$ to $|b\rangle$, the atom loses angular momentum. Using either an accepted symbol or a number with correct units, specify the quantity of angular momentum lost by the atom in the transition.

(f) Suppose the $|c\rangle$ to $|b\rangle$ transition is stimulated by a laser beam that is circularly polarized and propagating along the z direction. When the atom loses angular momentum, the angular momentum must be increased somewhere else. Where does the angular momentum go? (In other words, what gains angular momentum?) Be as specific as possible.

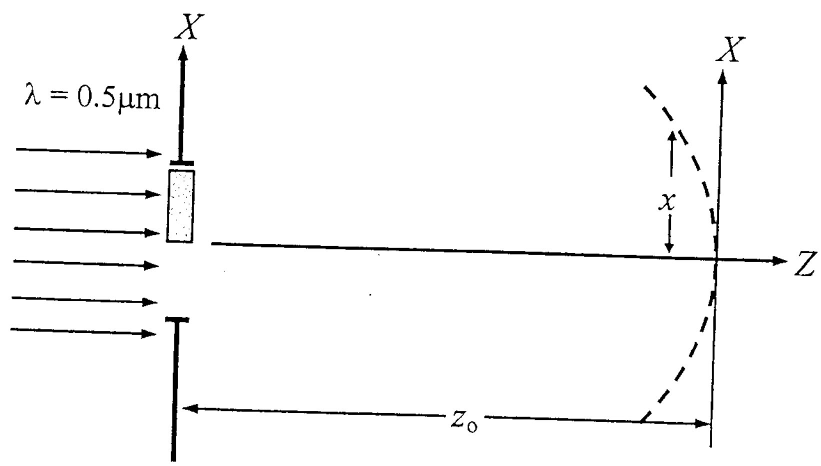
(g) If the $|c\rangle$ to $|b\rangle$ transition occurs by *spontaneous* emission, is total angular momentum (of the atom and surrounding fields) conserved in the process?

(h) State $|b\rangle$ can undergo further decay (to state $|a\rangle$, say) by emission of light. The $|b\rangle$ to $|a\rangle$ transition has a natural linewidth of $\Gamma = (2\pi) \times 99$ MHz. What is the natural lifetime τ of state $|b\rangle$?

(i) Write an expression for the probability vs. time t that a hydrogen atom in free space (and with no laser beams around) would be found in state $|b\rangle$, letting time t=0 be the instant the atom falls from $|c\rangle$ to $|b\rangle$. If you use symbols that have not been defined elsewhere in this problem, be sure to define them.

(j) If a narrow linewidth tunable laser is used to spectroscopically probe the energy level structure of this (or any other) atom, the $|n,l,m\rangle$ states are seen to give an incorrect description of atomic energy level structure. List 2 physical quantities that must be considered in calculating the actual atomic structure.

The slit aperture in the figure has a width of 2.0 mm along the X-axis. A collimated, monochromatic beam of light is incident on this aperture ($\lambda = 0.5\mu m$). The upper half of the aperture contains a transparent plate that acts as a 180° phase-shifter (i.e., it delays the incident rays by half a wavelength.) Write the functional form of the light's complex-amplitude distribution $a(x, z=0)$ immediately after the aperture. What is the far-field distribution $a(x, z=z_o)$ on a cylindrical surface centered on the aperture at $z = z_o$?



Diffraction with a pupil filter varying as x

a. Draw a graph of the Fourier transform of $sin(2\pi x)$.

b. Draw a graph of the Fourier transform, $F(\omega)$, of the function $f(x)=x$ and provide a concise description of $F(\omega)$. (See the endnote for a way to determine this Fourier transform.)

c. When the $F(\omega)$ of part b is convolved with another function $G(\omega)$, what basic calculus operation is performed on $G(\omega)$?

d. Consider a diffraction limited monochromatic imaging system with a square aperture, $P(x,y)$ limiting the exit pupil extending over the range $-1<x<1$ and $-1<y<1$ in arbitrary units,

$$P(x,y)=\Pi\left(\frac{x}{2}\right)\Pi\left(\frac{y}{2}\right).$$

Provide the equation for the amplitude point spread function (image amplitude) in the focal plane for this square aperture.

Now place a transmissive phase/amplitude filter in the exit pupil whose amplitude transmittance, $Q(x,y)$ varies as x in the x-direction,

$$Q(x,y)=x\,\Pi\left(\frac{x}{2}\right)\Pi\left(\frac{y}{2}\right).$$

So along a vertical line down the center of the pupil, the amplitude transmittance is zero; the amplitude transmittance is positive on the right side and negative on the left side of the pupil increasing to a transmittance of 1 along the right and -1 along the left edges of the pupil.

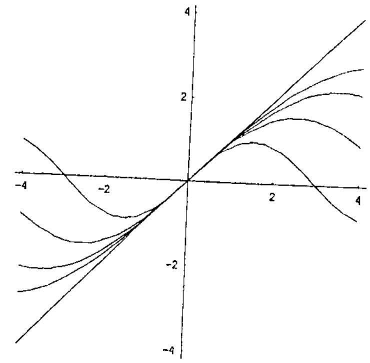
e. Draw crossections of the amplitude point spread function in the image plane along the x-axis and y-axis. How has the amplitude transfer function of the square aperture been modified?

f. Discuss issues in making such a $Q(x,y)$ filter. How might the filter be fabricated? How could the negative amplitude transmittance for the region $x<0$ be obtained? What would the intensity transmittance of the x-filter be?

Endnote: The Fourier transform of $f(x)=x$ can be determined in several ways as the limit of different series of Fourier transforms. Here is one of the simpler ways. Consider the function $2\pi L * \sin(2\pi x/2\pi L)$ as the period of the sinusoid, L, becomes infinitely long and the amplitude, also L, becomes infinitely large,

(continued next page)

Continuation of **Diffraction with a pupil filter varying as x**



$$F(\omega) = \lim_{L\to\infty} \prod \mathcal{F}\left\{2\pi L \sin\left(\frac{2\pi x}{2\pi L}\right)\right\}.$$

In the limit as L→∞, this series of sine functions of increasing amplitude and increasing period approaches the function x, so the Fourier transform of this series approaches the desired Fourier transform, $F(\omega)$. Only a graph of $F(\omega)$ is requested.

A binary optical signal with a data rate of 40 Gigabits per second and a wavelength of 1.55 μm is transmitted in a standard, single mode optical fiber. Estimate the maximum distance the optical signal can travel before pulse spreading becomes a significant impairment.

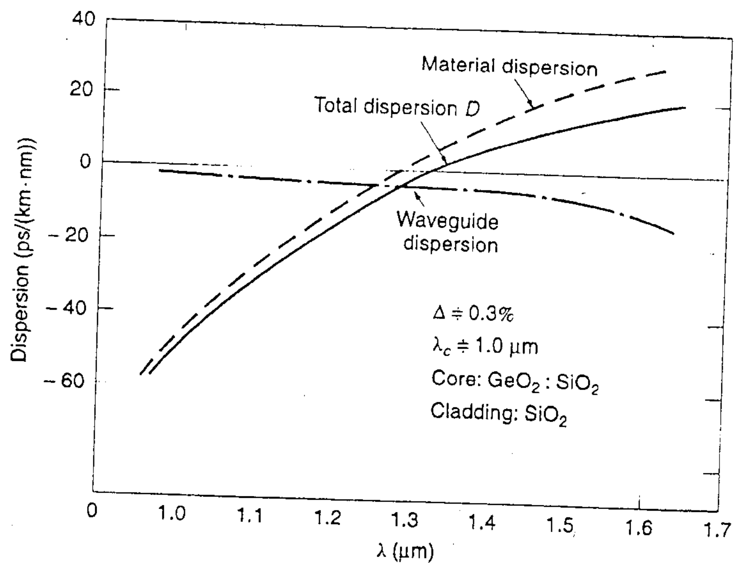


Figure 11.9 Dispersion of a single mode fiber. (After D. B. Keck [12].)

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$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\phi$

$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

$\nabla \cdot (\nabla \times \mathbf{F}) = 0$

$\nabla \times (\phi\mathbf{F}) = \phi(\nabla \times \mathbf{F}) + \nabla\phi \times \mathbf{F}$

$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

$\nabla \times \nabla\phi = 0$

$\oint_S (\mathbf{F} \cdot \mathbf{n})\, da = \int_V (\nabla \cdot \mathbf{F})\, d^3x$

$\oint_C \mathbf{F} \cdot d\ell = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da$

$\oint_S \phi\mathbf{n}\, da = \int_V \nabla\phi\, d^3x$

$\oint_S \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_V [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

$\oint_S (\mathbf{n} \times \mathbf{F})\, da = \int_V (\nabla \times \mathbf{F})\, d^3x$

Consider a linear, isotropic, and homogeneous medium for which the electric displacement $\vec{D}(t)$ can be expressed in terms of the time history of the electric field $\vec{E}(t)$ as

$$\vec{D}(t) = \epsilon_0 \vec{E}(t) + \int_{-\infty}^{\infty} d\tau \mathcal{R}(\tau) \vec{E}(t-\tau), \tag{1}$$

with $\mathcal{R}(\tau)$ the linear response function, and the spatial dependence of the fields has been suppressed for simplicity in notation.

(a) What condition must be imposed on the linear response function $\mathcal{R}(\tau)$ so that optical response described by Eq. (1) is causal? (1 points)

(b) By considering a monochromatic electric field of frequency ω show that Eq. (1) leads to the frequency-dependent electric displacement $\vec{D}(\omega) = \epsilon(\omega)\vec{E}(\omega)$, and derive an expression for the frequency-dependent dielectric constant $\epsilon(\omega)$ in terms of the linear response function. (3 points)

(c) Using the results from part (b) calculate the frequency dependence of the dielectric constant $\epsilon(\omega)$ for an instantaneously responding medium with $\mathcal{R}(\tau) = C\delta(\tau)$, C being a constant. (2 points)

(d) Next consider the linear response function $\mathcal{R}(\tau) = C\gamma\theta(\tau)\exp(-\gamma\tau)$, with $\gamma > 0$ and C constants, and $\theta(\tau)$ the Heaviside function which is zero for negative arguments and unity for positive arguments. Using the results from part (b) show that the corresponding dielectric constant is $\epsilon(\omega) = \epsilon_0 + C\gamma/(\gamma - i\omega)$. (3 points)

(e) In what frequency limit can the frequency-dependent dielectric constant from part (d) be approximated as arising from an instantaneously responding medium as in part (c)? (1 points)

The Telegrapher's equation for propagation of an electric field $\vec{E}(\vec{r},t)$ in a conducting medium of background refractive-index n occupying the half-space $z > 0$ is

$$\nabla^2\vec{E} - \nabla(\nabla\cdot\vec{E}) - \frac{n^2}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \mu_0\sigma\frac{\partial\vec{E}}{\partial t} = 0,$$

where $\sigma > 0$ is the conductivity.

(a) By considering a monochromatic plane-wave electric field

$$\vec{E}(\vec{r},t) = \frac{\hat{\mathrm{e}}}{2}\left[E_t e^{i(Kz-\omega t)} + c.c.\right], \quad z > 0$$

show that the complex propagation wavevector $K = k_r + i\alpha/2$, with k_r and α real, obeys $K^2 = n^2\omega^2/c^2 + i\omega\mu_0\sigma$, stating any assumptions made. (3 points)

(b) Next assume that a field is incident from vacuum onto the conducting medium from the half-space $z < 0$ and that some is also reflected

$$\vec{E}(\vec{r},t) = \frac{\hat{\mathrm{e}}}{2}\left[(E_i e^{i\omega z/c} + E_r e^{-i\omega z/c})e^{-i\omega t} + c.c.\right], \quad z < 0.$$

By demanding that the field and its first z-derivative be continuous at the interface at $z = 0$ derive an expression for the field reflectivity coefficient $r = E_r/E_i$. (4 points)

(c) Using the results from part (b), find approximate expressions for the field reflectivity coefficient r in the two limits $\sigma \to 0$ and $\sigma \to \infty$. (3 points)

Problem

a) Sketch the attenuation in dB/km for a typical silica optical fiber (used in modern Telecommunications) as a function of wavelength within the wavelength range of 800 nm - 1800 nm. The sketch should show the distinct features of the attenuation spectrum and give an approximate value for the minimum attenuation.

(5 points)

b) Discuss the two main intrinsic attenuation mechanisms fundamentally limiting the minimum attenuation in a typical single-mode silica optical fiber. Discuss and identify in your graph the major extrinsic cause for attenuation.

(5points)

Problem

a) Briefly discuss the different types of dispersion mechanisms in single mode fibers. **(3 points)**

b) The dispersion parameter $\boldsymbol{D}$ in single mode fibers is defined as:

$$D = -\frac{\lambda}{c}\frac{\partial^2 n_{core}}{\partial \lambda^2} - \frac{n_{core}\Delta}{c\lambda} V \frac{d^2(Vb)}{dV^2}$$

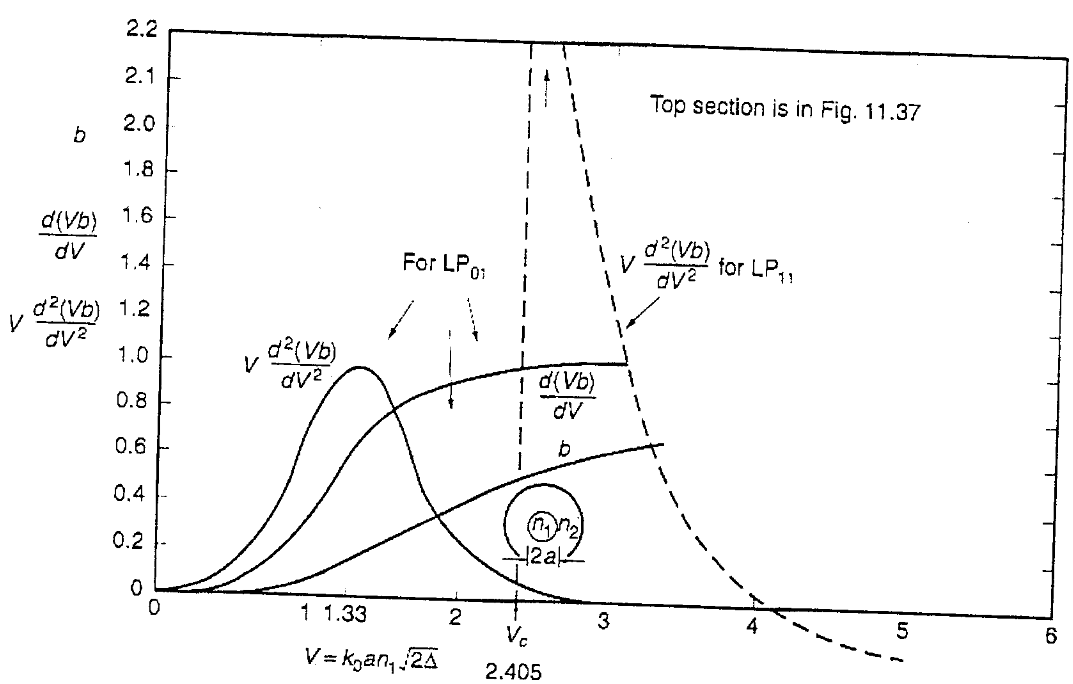
What do the two terms in the dispersion parameter represent? **(2 points)**

c) Design a step-index single mode fiber (**i.e. choose the core radius and cladding index**) resulting in zero dispersion at 1550 nm wavelength. Assume a core refractive index of n_{core} = 1.458 and use a value for Δ less than 0.009. The material dispersion at 1550 nm wavelength is ~ 20 ps/(nm-km).

(5 points)

Recall: $\quad \Delta = \dfrac{n_{core} - n_{clad}}{n_{core}}$

You may use the graph below.



A physical variable that fluctuates randomly as a function of time can be described by a stochastic process $f(t)$.

a) Provide a definition for the mean of this process $\overline{f}(t)$

b) Provide a definition for the autocorrelation function $R(t,t')$ and the autocovariance function $K(t,t')$

c) Explain what we mean when we say that this process is stationary

d) What conditions are placed on $\overline{f}(t)$, $R(t,t')$ and $K(t,t')$ by the stationarity assumption?

e) Suppose that this process is stationary. Provide a definition for the power spectral density (PSD)

The Wiener-Khinchin theorem relates the PSD to the expected value of the square magnitude of the periodogram.

f) Provide a definition of the periodogram

g) State the Wiener-Khinchin theorem

h) Discuss how we might estimate the PSD from measurements of $f(t)$

Consider a simplified Stern-Gerlach experiment in which a spin-1/2 particle is subjected to a spin-dependent force along the x-axis. At the start of the interaction between the particle and the Stern-Gerlach apparatus the overall quantum state is

$$|\Psi(0)\rangle = (\alpha|+\rangle + \beta|-\rangle)|\varphi\rangle,$$

$$\text{where } \sigma_z|\pm\rangle = \pm|\pm\rangle \text{ and } \varphi(x) = \langle x|\varphi\rangle = \begin{cases} 1/\sqrt{d} & \text{for } -d/2 \le x \le d/2 \\ 0 & \text{elsewhere} \end{cases}$$

Inside the apparatus the wavepackets corresponding to the $|\pm\rangle$ states are translated by $\pm\lambda$ along the x-axis. If we ignore spreading of the wavepacket the state of the particle as it emerges from the apparatus is then

$$|\Psi(\lambda)\rangle = \alpha|+\rangle S(\lambda)|\varphi\rangle + \beta|-\rangle S(-\lambda)|\varphi\rangle,$$

where $S(\lambda) = e^{-i\lambda P/\hbar}$ is the translation operator. In the following we examine how a measurement of the particle position can be used to measure the spin state.

(1) Find the conditional probability densities $P(x\,|+)$ and $P(x\,|-)$, i. e. the probability of measuring a position in the interval $[x, x+dx]$, given that a prior measurement of σ_z has resulted in an outcome +1 or −1. Find also the probability density associated with a measurement of the position, for an arbitrary value of the translation. Sketch all three probability densities as a function of x. (35%)

(2) Use Bayes rule, $P(h\,|\,g) = P(g\,|\,h)P(h)/P(g)$, to find the conditional probabilities $P(+\,|\,x)$ and $P(-\,|\,x)$ for an arbitrary value of λ. At what critical translation, $\lambda = \lambda_c$, does a measurement of X always correspond to an unambiguous measurement of σ_z? What is the general form of the particle state after such an unambiguous result? (40%)

(3) Now assume that $\lambda < \lambda_c$, in which case there exist a range of possible outcomes for a measurement of X which do not give any information about the spin state. What is the quantum state of the particle following such a measurement outcome? (25%)

WRITTEN PRELIM EXAM – SECOND DAY

Fall 2004

September 22, 2004
8:30 a.m. to 12:00 p.m.

Answer all six questions. Start each answer on a new page.

In the upper right hand corner of each sheet you hand in, put your code number (NO NAMES PLEASE) and the problem number. Staple together all sheets for a given problem.

Insert your answers into the manila envelope supplied and note which problems you have answered on the outside of the envelope.

The following are some helpful items:

$h = 6.625 \times 10^{-34}\ \mathrm{J \cdot s} = 4.134 \times 10^{-15}\ \mathrm{eV \cdot s}$

$e = 1.6 \times 10^{-19}\ \mathrm{C}$

$c = 3.0 \times 10^{8}\ \mathrm{m/s}$

$k_B = 1.38 \times 10^{-23}\ \mathrm{J/K}$

$\sigma = 5.67 \times 10^{-8}\ \mathrm{W/K^4 \cdot m^2}$

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}$

$\mu_0 = 1.26 \times 10^{-6}\ \mathrm{H/m}$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = 2 \cos^2 A - 1$

$\cos 2A = 1 - 2 \sin^2 A$

$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$

$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$\nabla\phi\psi = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\phi$

$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

$\nabla \cdot (\nabla \times \mathbf{F}) = 0$

$\nabla \times (\phi\mathbf{F}) = \phi(\nabla \times \mathbf{F}) + \nabla\phi \times \mathbf{F}$

$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

$\nabla \times \nabla\phi = 0$

$\oint_S (\mathbf{F} \cdot \mathbf{n})\, da = \int_V (\nabla \cdot \mathbf{F})\, d^3x$

$\oint_C \mathbf{F} \cdot d\ell = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da$

$\oint_S \phi\mathbf{n}\, da = \int_V \nabla\phi\, d^3x$

$\oint_S \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_V [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

$\oint_S (\mathbf{n} \times \mathbf{F})\, da = \int_V (\nabla \times \mathbf{F})\, d^3x$

A quantum system consists of an electron that is confined to move in a circle of constant radius a in the x-y plane, so that the position $\mathbf{r} = (x, y) = (a\cos\theta, a\sin\theta)$. The confining potential does not depend on the angle θ.

(1) Write down the operator corresponding to the orbital angular momentum L_z and the total energy H in the θ-representation (in terms of θ and derivatives with respect to θ). (30%)

Hint: it may help you find the θ-representation of the tangential momentum operator if you imagine "unwrapping" the circle and compare to the usual linear momentum.

(2) Find the common eigenfunctions of L_z and H, and the corresponding eigenvalues. What is the degeneracy of the eigenvalues? (30%)

Now imagine that an electric field is applied across the channel in which the electron is moving, so that it sees a purely radial electric field $\mathbf{E} = E_0\,\mathbf{r}/r$. In analogy to the problem of fine structure in the Hydrogen atom, the moving electron will see a magnetic field $\mathbf{B} = -(\mathbf{v}\times\mathbf{E})/c^2$, which will couple to the electron magnetic moment $\boldsymbol{\mu}_S = q\mathbf{S}/m_e$ and add an extra term $W = -\boldsymbol{\mu}_S \cdot \mathbf{B}$ to the Hamiltonian.

(3) Calculate the new eigenenergies of the system in the presence of the perturbation W. What is the degeneracy of the perturbed energy levels? (40%)

(There are 5 parts, each is worth 20 percent.)

We consider a two-level atom with excited level $|e\rangle$ and ground level $|g\rangle$ separated by the energy $\hbar\omega$. This atom is dipole-coupled to a quantized single-mode electromagnetic field of frequency ω. This system is described by the Hamiltonian

$$H = \hbar\omega|e\rangle\langle e| + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g\left(\hat{a}|e\rangle\langle g| + H.c.\right).$$

I.

What are the eigenstates and eigenenergies of this Hamiltonian for the case $g = 0$?

II.

Find the eigenstates and eigenenergies of H for the case $g \neq 0$. (Hint: you may want to use the fact that $\sin^2\theta + \cos^2\theta = 1$, although there are other ways to proceed.)

III.

Find the dynamics of the system in case its initial state is $|\psi(0)\rangle = |e, n\rangle$, where $|n\rangle$ is a number (Fock) state. What happens for $n = 0$? What happens if the initial condition is $|g, n = 0\rangle$ instead? Explain the physics behind this result.

IV.

Assume now that the field that the atom is interacting with is a *fermionic* field instead of a bosonic field. Remember that in that case, the operators a and $a^\dagger$ anticommute rather than commute, $[\hat{a}, \hat{a}^\dagger]_+ = 1$, $[\hat{a}, \hat{a}]_+ = [\hat{a}^\dagger, \hat{a}^\dagger]_+ = 0$. Show that the Pauli Exclusion Principle follows directly from these anticommutation relations.

V.

Assuming that the initial state of the two-level atom coupled to this fermionic field by the interaction

$$\hbar g\left(\hat{a}|e\rangle\langle g| + H.c.\right)$$

is $|\psi(0)\rangle = |e, n = 1\rangle$. What is the probability for the atom to be in the ground state at some later time t? Explain your result in physical terms.

Each of the following parts is a verbal description of the pupil function in a unit-magnification ($p = q = 2f$) imaging system. For each part, sketch and discuss the coherent transfer function, the coherent PSF, the incoherent transfer function, and the incoherent PSF. Qualitative answers are desired, not detailed derivations, but all axes should be labelled clearly and critical dimensions (such as cutoff frequencies) should be specified.

(a) A rectangular slit of dimensions $L_x \times L_y$.

(b) Same as (a) but in the limit $L_x \to 0$.

(c) Two slits of dimensions $L_x \times L_y$, centered at $(\pm x_0, 0)$, $x_0 > L_x$.

(d) A square-wave grating of period ℓ placed over the aperture of part (a), with $\ell << L_x$ and the grating bars parallel to the y axis.

(e) A circular aperture of diameter D with a central obscuration of diameter d.

Note: All parts carry equal weight.

Principal planes and ABCD matrices.

The ABCD matrix of paraxial optics relates a ray height, h, and angle, u, in one plane to the height and angle in another plane

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} h_j \\ u_j \end{pmatrix} = \begin{pmatrix} h_{j+1} \\ u_{j+1} \end{pmatrix},$$

a. Provide the ABCD matrix for translation by L. Assume air throughout this problem. (2 points)

b. For an ABCD matrix between a paraxial object plane and paraxial image plane, which element must be zero? (2 points)

c. For the matrix of part b, determine the magnification, M, in terms of the matrix elements. (2 points)

Consider the following cases of the ABCD matrix between a paraxial object plane and paraxial image plane:

d. Which matrix elements are zero for an imaging system with the image plane at the back focal plane? (2 points)

e. Which matrix elements are zero for an afocal imaging system? (2 points)

Determine the low-frequency dispersion $\omega(k)$ of the lower polariton branch in a solid characterized by the dielectric function

$$\varepsilon(\omega) = \varepsilon_\infty - \frac{\omega_{pl}^2}{\omega^2 - \omega_0^2}$$

(Gaussian units are used in this problem). Express your result in terms of the speed of light c, the wave vector modulus k ($= |\vec{k}|$) and the zero-frequency dielectric constant ε_0. Proceed strictly according to the following instructions.

Instructions: First, derive the general polariton dispersion relation in terms of ω, k, c and $\varepsilon(\vec{k}, \omega)$ starting from the Fourier transform of the following Maxwell equation for the transverse E-field

$$\left\{ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right\} \vec{E}(\vec{r}, t) = -\frac{4\pi}{c^2} \frac{\partial^2}{\partial t^2} \vec{P}(\vec{r}, t)$$

It is sufficient if you state the result of the Fourier transform of this equation (without any intermediate steps). Using linear response theory for a spatially homogeneous medium to specify $\vec{P}(\vec{k}, \omega)$ in terms of the light field, you can easily derive the general dispersion relation. Finally, use the specific model for the dielectric function given above, evaluated in the low-frequency limit (i.e. you may neglect terms of the order $\frac{\omega}{\omega_0}$), to derive the low-frequency limit of the dispersion of the lower polariton branch.

Consider a semiconductor quantum well of thickness L_z with infinite barriers. Determine the lowest intersubband transition energy of the conduction subbands. Express your results in terms of the thickness and the effective mass m_e of the conduction band (assumed to be parabolic and isotropic). Proceed strictly according to the instructions.

Instructions: Assume the quantum well to be symmetrically placed around $z = 0$. Specify the Schroedinger equation for the conduction envelope wavefunction $\xi(z)$. To make things simple, assume that the eigenfunctions are either *cos* or *sin* functions, but not superpositions thereof. From the solution of the Schroedinger equation, obtain the desired intersubband transition energy.