TRACK I
DAY 1

WRITTEN PRELIM EXAM – FIRST DAY

Fall 2005

September 20, 2005
8:30 a.m. to 12:00 p.m.

Answer all six questions. Start each answer on a new page.

In the upper right hand corner of each sheet you hand in, put your code number (NO NAMES PLEASE) and the problem number. Staple together all sheets for a given problem.

Insert your answers into the manila envelope supplied and note which problems you have answered on the outside of the envelope.

The following are some helpful items:

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$k_B = 1.38 \times 10^{-23}\ \mathrm{J/K}$

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$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}$

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$\sinh x = \frac{1}{2}(e^x - e^{-x})$

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$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$\nabla\phi\psi = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\phi$

$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

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Fall 2005 Prelims
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Question 1

A biconvex thick lens in air has the following prescription and is used with a distant object:

R1 = 250 mm	$n_F = 1.828$
R2 = -500 mm	$n_d = 1.805$
t = 50 mm	$n_C = 1.796$

a) (70%) Use Gaussian reduction to determine the longitudinal chromatic focus shift of the lens.

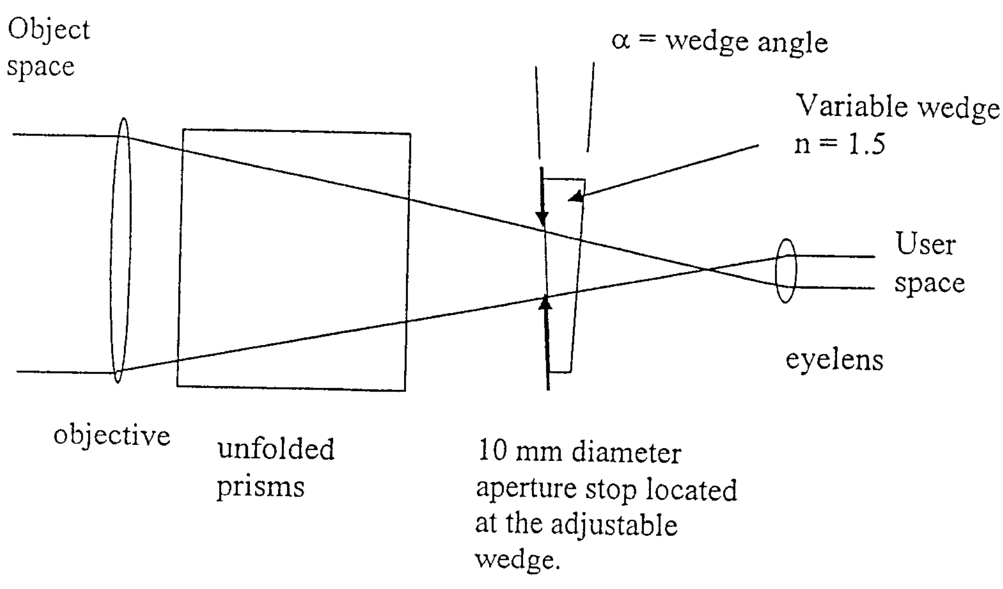
b) (30%) The lens diameter is 20 mm. Sketch the images at d-focus of a distant point source for F-light and for C-light. Provide the approximate spot sizes at these two wavelengths. Assume no other aberrations.

Fall 2005 Prelims
Track I, Day 1
Question 2

Image stabilization
Canon sells binoculars that actively stabilize the image using a variable wedge. The angular accelerations are measured with sensors and a correction is applied by changing the wedge angle of a fluid filled cavity. You can treat the active element as a wedge of refractive material with variable angle.

For the case of 20 x 50 binoculars, calculate the change in the wedge required to fully compensate the effect of tilting the binoculars by 1 mrad.

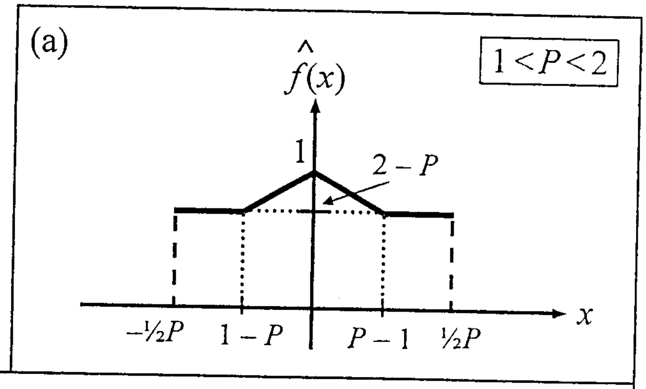
a) (15) Give the diameters of the entrance pupil, the exit pupil, and the magnifying power
b) (30) If the binoculars are tilted by 1 mrad while pointing at a fixed object far away, calculate the angular motion as observed by the user.
c) (30) Show the relationship between a change in the wedge angle $\Delta\alpha$ and the deviation it causes in user space when the system is pointed at a fixed distant object. A 10 mm diameter aperture at the wedge defines the system aperture stop.
d) (25) Calculate the required wedge to correct for the 1 mrad tilt of the binoculars



Fall 2005 Prelims
Track 1, Day 1
Question 3

(2 pts) **Problem 1a)** Find the Fourier transform $\hat{F}(\sigma)$ of the function $\hat{f}(x)$ in Fig. 1(a).

(2 pts) **b)** The periodic function $f(x)$ in Fig. 1(b) is a periodic repetition of $\hat{f}(x)$. What is the Fourier transform $F(\sigma)$ of $f(x)$?



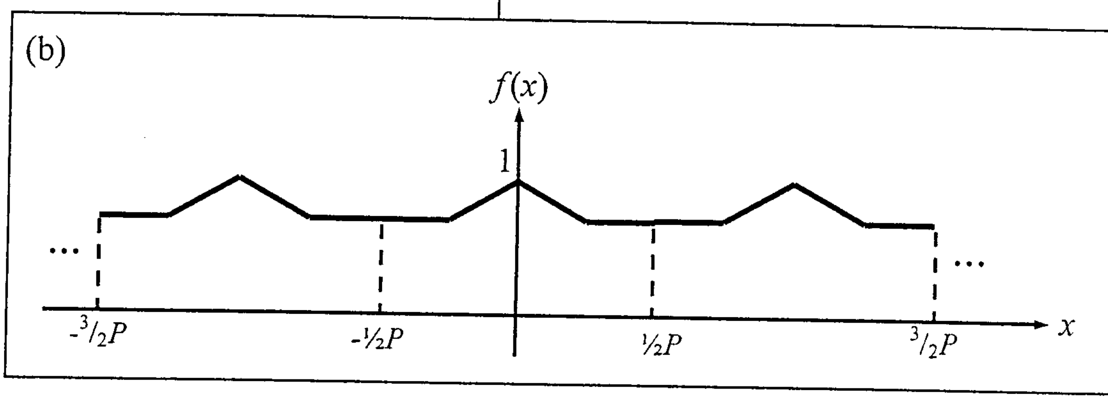


Figure 1

Problem 2) Consider a linear shift-invariant system with input $f(x)$, output $g(x)$.

(2 pts) **a)** Show that if $\frac{d}{dx} f(x)$ is input, $\frac{d}{dx} g(x)$ will be output.

(2 pts) **b)** Let $f(x) = step(x)$ be used as input, and assume that the corresponding output is

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\alpha x} & x \geq 0 \end{cases}$$

Determine the impulse response $h(x)$ of the system.

(2 pts) **c)** Again let $f(x) = step(x)$ be used as input. This time assume the corresponding output is

$$g(x) = \begin{cases} 0 & x < 0 \\ e^{-\alpha x} & x \geq 0 \end{cases}$$

Determine the impulse response $h(x)$ of the system.

Note: You may assume that α is a positive, real number.

Fall 2005 Prelims
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Question 4

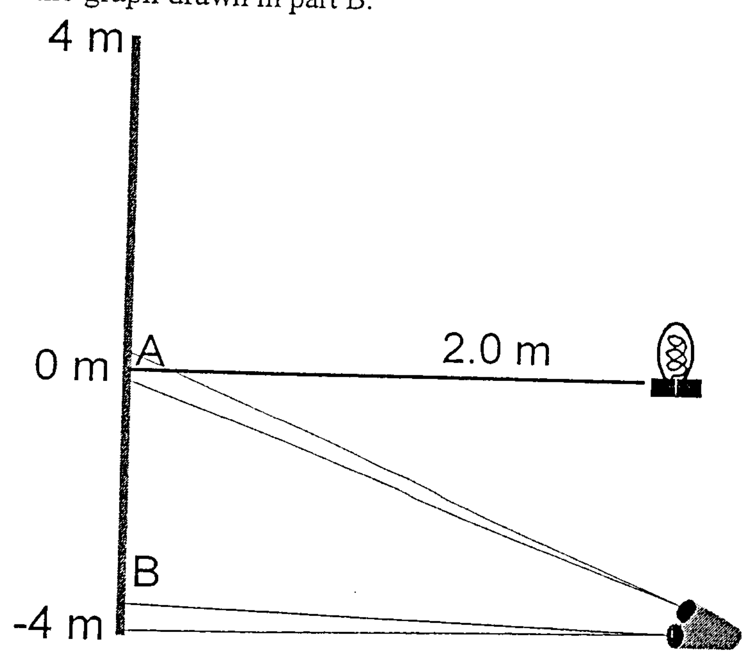
A lamp source that can be considered to be small enough so as to be approximated by a point source is placed 2.0 m from the center of a wall that is a total of 8 m in length. The lamp output is 3142 W and the accompanying figure illustrates the geometry.

A) A radiometer is placed at one edge of the wall (position -4 m) at a distance of 2.0 m from the wall. Compute the radiant flux through the radiometer's detector when the radiometer is pointed at the center of the wall (at point A corresponding to the 0-m position). The reflectance of the wall in this direction is 0.70. The radiometer is a simple tube based radiometer with a circular detector that is 0.02 cm in radius at the end of a 20-cm long tube with a 2-cm radius aperture.

B) Draw a graph of the radiance from the wall as seen by the radiometer as it is scanned along the wall from the +3 to -3 m position. State all assumptions.

C) A lens with a 20-cm focal length is placed on the front of the radiometer. What is the radiant flux for the geometry in part A?

D) The radiant flux computed in part C must be doubled to achieve an appropriate signal to noise ratio. Describe one approach to doubling the radiant flux and the impact this would have on the graph drawn in part B.

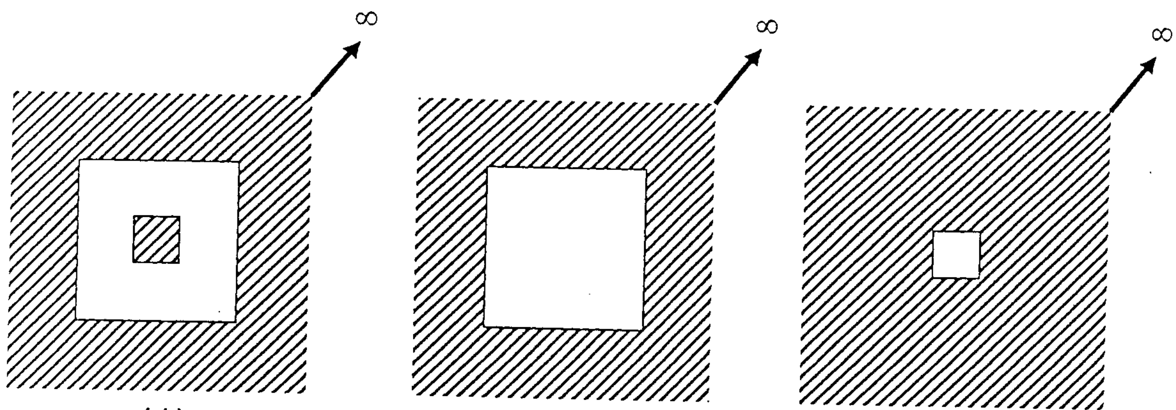


Fall 2005 Prelims

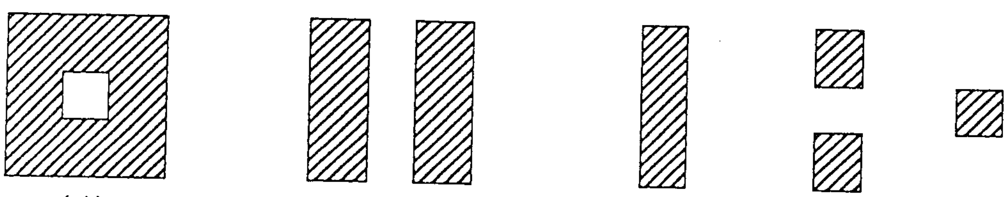
Track 1, Day 1

Question 5

The diffracted amplitude and phase from each aperture shown below is measured at a distance z from the aperture. The field measurements are denoted $f_i(x,y)$, where $i = 1, 2, \ldots 10$. Find four combinations of apertures such that $f_i(x,y) = f_j(x,y) + f_k(x,y)$, where $i \neq j \neq k$. Assume identical measurement conditions for each aperture. (Shading indicates an opaque region.)



(1) (2) (3)



(4) (5) (6) (7) (8)

(9) transparent

(10) opaque

$f_i(x,y) = f_j(x,y) + f_k(x,y)$

i	j	k

Fall 2005 Prelims
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Question 6

(a) Write down the macroscopic Maxwell equations in the MKS system of units including contributions from both bound and free charges (2).

(b) Derive the wave equation for propagation of the electric field $\vec{E}(\vec{r},t)$ in a non-magnetic, non-conducting medium of constant refractive-index n starting from the Maxwell equations in part (a), making all assumptions clear (2).

(c) The wave equation in part (b) has a solution in the form of a travelling wave propagating along the z-axis, $\vec{E}(\vec{r},t) = \vec{e} E_0 f(\xi)$, with $\xi = (kz - \omega t + \theta)$, $\vec{e}$ the unit polarization vector, E_0 the electric field amplitude, and θ the phase. Using this solution derive the dispersion relation for light propagation in the medium (3), and determine the group velocity of the solution (1).

(d) Show that the Maxwell equations impose a constraint on the choice of unit polarization vector $\vec{e}$ for the solution in part (c) (2).

TRACK I
DAY 2

WRITTEN PRELIM EXAM – SECOND DAY

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Fall 2005 Prelims
Track 1, Day 2
Question 8

Let us say we have a linear, shift-invariant imaging system which maps a one-dimensional object $o(x)$ to an image $i(x)$ via a convolution with a point-spread function $s(x)$. That is,

$$i(x) = o(x) \star s(x),$$

where $\star$ represents a one-dimensional convolution. Let us now assume that the object $o(x)$ is a wide-sense stationary stochastic process with mean m_o and autocorrelation $R_o(\Delta x)$. For the sake of simplicity, assume that $i(x)$, $o(x)$, and $s(x)$ are all real and that this imaging system is a gain 1 system, i.e., $\int_{-\infty}^{\infty} s(x) = 1$.

1. (30%) Write the statistical definitions of the mean and autocorrelation of the process $o(x)$. That is, what are m_o and $R_o(\Delta x)$ in terms of expectations over the process $o(x)$?
2. (30%) Because $o(x)$ is a stochastic process, $i(x)$ must also be a stochastic process. Write the mean and autocorrelation of $i(x)$ in terms of m_o, $R_o(\Delta x)$, and $s(x)$.
3. (5%) Is $i(x)$ also wide-sense stationary?
4. (30%) Now, let us account for detector noise by assuming that we measure a noisy image $i'(x) = o(x) \star s(x) + n(x)$, where $n(x)$ is a wide-sense stationary stochastic process with mean 0 and autocorrelation $R_n(\Delta x)$. Assume that $n(x)$ is independent of $o(x)$. What are the mean and autocorrelation of $i'(x)$?
5. (5%) Is $i'(x)$ wide-sense stationary?

Fall 2005 Prelims
Track 1, Day 2
Question 9

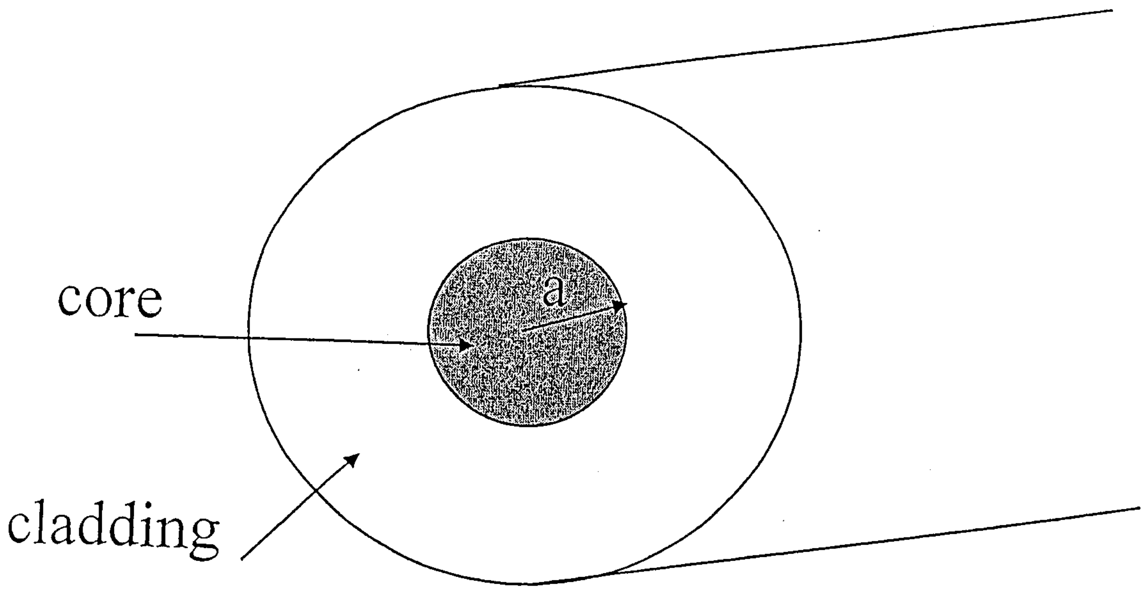
Speckle intensity v is known to follow a negative exponential probability density function. That is,

$$p_I(v) = \frac{1}{\mu} \exp(-v/\mu); \quad v \geq 0,$$

where μ is the mean intensity or $\langle v \rangle$.

1. (30%) Derive the characteristic function $\phi_I(\omega)$ for the random variable v.
2. (30%) Use this characteristic function to prove that the $\langle v \rangle = \mu$. (Hint: This involves differentiating $\phi_I(\omega)$ with respect to ω evaluated at $\omega = 0$.)
3. (30%) Imagine that we took two intensity measurements v_1 and v_2 at two separate times. What is the characteristic function of the sum of v_1 and v_2? You can assume that the random variables v_1 and v_2 are independent from one another.
4. (10%) Now assume we took many independent measurements v_i where i goes from 1 to a large number (say 100). Approximately what probability distribution does $z = \sum_{i=1}^{100} v_i$ follow? (No derivation necessary.)

Fall 2005 Prelims
Track I, Day 2
Question 10



Consider light propagation in a step-index optical fiber (pictured above) with a core radius a, core refractive index n_{core} and cladding refractive index n_{clad}. The "characteristic equation" for a step index fiber can be written as:

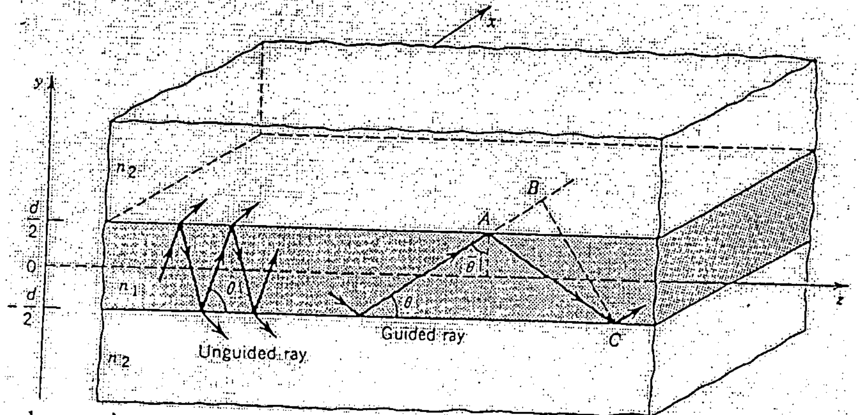
$$\frac{\beta^2\nu^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2=\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

a) In analyzing step index optical fibers a *weakly guiding* approximation is often justified. Explain what it means and show how the "characteristic equation" simplifies to a form that does not explicitly contain β (no need to use Bessel function relations). **(4 points)**

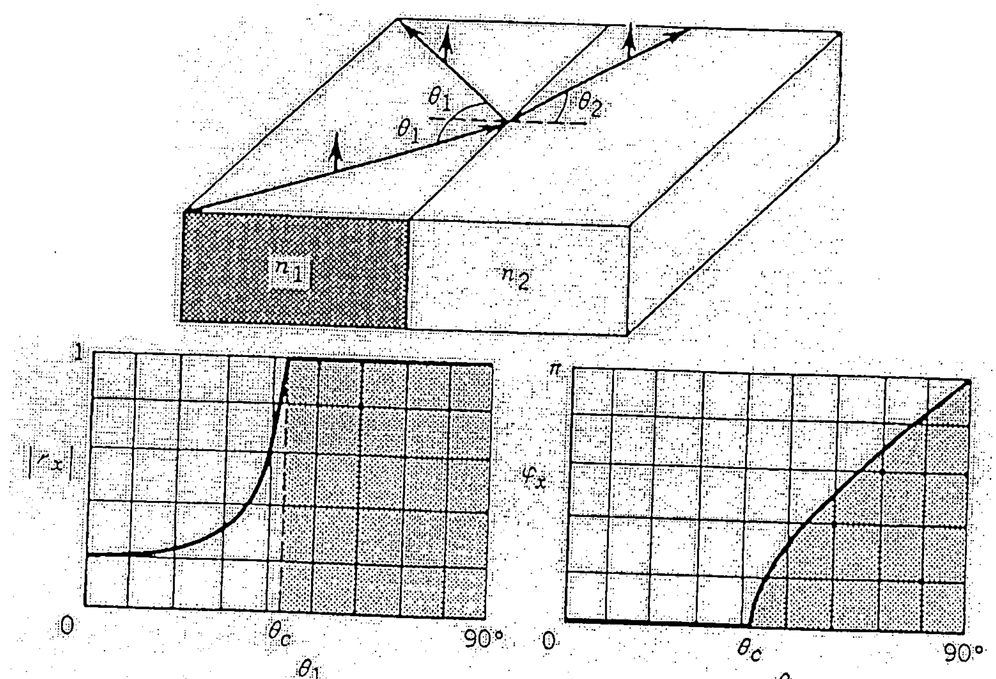
b) Explain the meaning of transverse modes in a step index fiber and show how the "characteristic equation" simplifies (no need to use Bessel function relations). **(4 points)**

c) With *weakly guiding* approximation, sketch the electric and magnetic field lines for the LP_{01} -mode in a cross section of a <u>single mode fiber</u> (indicate also the radial dependence of the fields). Briefly explain the degeneracy of the mode. **(2 points)**

Fall 2005 Prelims
Track I, Day 2
Question 11



The bouncing ray picture of guided waves in a planar slab waveguide.



The magnitude $|r_x|$ and phase ϕ_x of the reflection coefficient for internal reflection of a TE wave on a planar interface. The phase is given by the expression:

$$\tan\frac{\phi_x}{2} = \frac{\left(\sin^2\theta_1 - \sin^2\theta_c\right)^{1/2}}{\cos\theta_1}.$$

1. Find an implicit expression for the bouncing angles θ_m (see the bouncing ray picture above) for the guided waves in terms of the refractive indices n_1 and n_2, the core thickness d, and the vacuum wavelength λ_0. (The trig identity $\cos 2\theta = 1 - 2\sin^2\theta$ may be helpful.) **(7 points)**

2. Very briefly describe the origin of waveguide dispersion in the bouncing ray picture. **(3 points)**

Fall 2005 Prelims
Track I, Day 2
Question 12

Previous experiments have demonstrated the single-transverse mode propagation of ultra-cold atoms through a waveguide, in analogy with single-mode optical waveguiding. For example, atoms can be guided through a hollow core optical fiber as long as the atoms are kept away from the inner wall of the fiber using off-resonant laser light propagating in the cladding layer; an evanescent light field then creates a repulsive transverse potential (in the x and y directions). The light is tuned well above atomic resonance, so photon absorption and emission are negligible. Atoms can thus be longitudinally guided down the center of the hollow core. In this problem, we'll consider an atom of mass m propagating down a hollow fiber, and assume that the *total* potential energy near the fiber axis is $V(x,y) = (1/2)m\omega^2(x^2+y^2)$. **There is no longitudinal (z-direction) potential.** In the questions below, you will work only with the transverse modes of the guiding potential, thus this is a 2D problem and the z direction will be neglected. You will need to use some or all of the following:

$\hbar = 10^{-34}$ J·s,

$\sigma = \sqrt{\frac{\hbar}{m\omega}}$

$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$.

Answer the following questions. *Partial credit is available for partial answers and good guesses!*

(a) *(1.5 pts.)* Write a complete expression for the 2D (transverse) Hamiltonian $\hat{H}(x,y)$.

(b) *(2 pts.)* Let $\psi_0(x,y) = A \cdot e^{-x^2/(2\sigma^2)} \cdot e^{-y^2/(2\sigma^2)}$ be the expression for the ground energy eigenstate of $\hat{H}(x,y)$. Assuming that $\psi_0(x,y)$ is normalized in the usual way, solve for A in terms of σ and assume A is real and positive.

(c) *(2 pts.)* In terms of constants and variables used above, evaluate the energy eigenvalue E_0 corresponding to $\psi_0(x,y)$. This is the transverse energy of the state $\psi_0(x,y)$. If you are unsure how to answer or are running out of time, make a good guess and justify it. Use your answer as needed in the questions below.

(d) *(1 pt.)* Evaluate $\langle x \rangle$ for the state ψ_0.

(e) *(1 pt.)* Let R_c represent the classical turning point radius of $\psi_0(x,y)$. R_c is thus just a magnitude, i.e., a distance in the (x,y) plane away from the point $(x,y) = (0,0)$. Evaluate R_c in terms of σ. (HINT: let $r^2 = x^2 + y^2$.)

(f) *(1 pt.)* Let $r^2 = x^2 + y^2$. Provide a guess for $\langle r^2 \rangle$ for the ground state, but *do not* try to fully evaluate it mathematically. As long as the answer makes some sense and falls within reasonable limits, you will receive credit for your answer.

(g) *(1 pt.)* Write an expression in terms of r and R_c that is equivalent to the total probability of finding the atom *outside* of the classically allowed region. Do not try to evaluate this expression.

(h) *(0.5 pt.)* Suppose that the fiber is pinched along the x-direction, and instead of an atom being in the ground state, it is in the first excited transverse state (the state that is next higher in energy above the new pinched-fiber ground state). On an x,y plot, illustrate the probability density distribution, or spot pattern, for this state. (Hint: there is no state degeneracy to consider.)

TRACK II
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$\nabla \times (\phi\mathbf{F}) = \phi(\nabla \times \mathbf{F}) + \nabla\phi \times \mathbf{F}$

$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

$\nabla \times \nabla\phi = 0$

$\oint_S (\mathbf{F} \cdot \mathbf{n})\, da = \int_V (\nabla \cdot \mathbf{F})\, d^3x$

$\oint_C \mathbf{F} \cdot d\ell = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da$

$\oint_S \phi\mathbf{n}\, da = \int_V \nabla\phi\, d^3x$

$\oint_S \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_V [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

$\oint_S (\mathbf{n} \times \mathbf{F})\, da = \int_V (\nabla \times \mathbf{F})\, d^3x$

Fall 2005 Prelims
Track II, Day 1
Question 1

Consider the one-dimensional Langevin equation for $v(t)$:

$$\dot{v} = h(v) + \Gamma(t)$$

with the damping term given by

$$h(v) = -\frac{2\gamma v}{1 - v^2}$$

We make the usual assumptions about the Gaussian random process $\Gamma(t)$:

$$\langle \Gamma(t) \rangle = 0$$

and

$$\langle \Gamma(t) \Gamma(t') \rangle = 2\delta(t - t')$$

a) Write down the Fokker-Planck equation for the distribution function $W(v, t)$. (2 points)

b) Write down the differential equation for the equilibrium distribution $W(v)$. (2 points)

c) Solve for $W(v)$. There will be a normalization constant in your answer. (2 points)

d) Use the change of variable

$$u = \frac{1}{2}(1 + v)$$

to show that the distribution for u must be proportional to the beta distribution function

$$\frac{u^{a-1}(1-u)^{b-1}}{B(a, b)}$$

for some a and$b$, where $B(a, b)$ is the beta function. Note: You do not need to know what the beta function is. (2 points)

e) Now find an expression for the normalization constant in terms of the beta function. (2 points)

Fall 2005 Prelims
Track II, Day 1
Question 2

(a) Write down the macroscopic Maxwell equations in the MKS system of units including contributions from both bound and free charges (2).

(b) Derive the wave equation for propagation of the electric field $\vec{E}(\vec{r},t)$ in a non-magnetic, non-conducting medium of constant refractive-index n starting from the Maxwell equations in part (a), making all assumptions clear (2).

(c) The wave equation in part (b) has a solution in the form of a travelling wave propagating along the z-axis, $\vec{E}(\vec{r},t) = \vec{e}E_0 f(\xi)$, with $\xi = (kz - \omega t + \theta)$, $\vec{e}$ the unit polarization vector, E_0 the electric field amplitude, and θ the phase. Using this solution derive the dispersion relation for light propagation in the medium (3), and determine the group velocity of the solution (1).

(d) Show that the Maxwell equations impose a constraint on the choice of unit polarization vector $\vec{e}$ for the solution in part (c) (2).

Fall 2005 Prelims
Track II, Day 1
Question 3

(a) Explain in words what is meant by the term "dipole approximation" as used in connection with the interaction of optical fields with atoms. (2)

(b) The Lorentz oscillator model for the interaction between an electron oscillator and the electromagnetic field is

$$\left[\frac{d^2}{dt^2}+\gamma\frac{d}{dt}+\omega_0^2\right]\vec{x}(t)=\frac{e}{m}\vec{E}(t) \ .$$

Give a brief discussion of the physical meaning of the quantities $\vec{x}$, γ, and ω_0 (2), and explain which of these has its physical origin in the presence of a longitudinal electric field (1).

(c) For a monochromatic field $\vec{E}(t)=\left[\vec{E}(\omega)e^{-i\omega t}+c.c\right]/2$, and in the limit $\omega_0,\gamma \to 0$, show that the frequency-dependent index of refraction can be written as $n^2(\omega)= 1+\chi(\omega)=1-(\omega_{pl}/\omega)^2$ for a medium with a density N of Lorentz oscillators (2).

(d) Based on the results of part (c) explain why the medium becomes highly absorbing for frequencies $\omega<\omega_{pl}$ (1).

(e) What sort of medium is the Lorentz model describing in the limit employed in part (c) (1), and why do you think this is the case (1)?

Fall 2005 Prelims
Track II, Day 1
Question 4

(a) (2 points) Consider a two-electron atom such as He. For each of the two-electron configurations *nsn's* and *nsn'p*, what are the possible values for the quantum numbers l and s associated with the total orbital and total spin angular momenta, respectively. Give also the possible values of the quantum number j for the total (orbital plus spin) angular momentum of the two electron system. The notation $nln'l'$ ($l,l' = s,p,d$) indicates that there is one electron in the nl state, and another in the $n'l'$ state.

(b) (1 point) Give the possible values for l, s and j in the *nsns* configuration.

(c) (1 point) Consider a 2-level atom with a ground state $|g\rangle$ and excited state $|e\rangle$, interacting with a mode of an optical cavity whose frequency equals the atomic transition frequency, $\omega = (E_e - E_g)/\hbar$. Give an example of an entangled state of the composite system for which the total energy is $\hbar\omega$.

(d) (2 points) For the entangled state in (c) find the density operator that describes the atom separately from the cavity.

(e) (2 points) The spin component of an electron along the z axis has been reliably established to be +1/2. (i) What is the probability $w(+1/2)$ that the projection of the spin on an axis z' making an angle θ with the z axis will have the value +1/2? (ii) What is the probability $w(-1/2)$ that it will have the value -1/2?

(f) (3 points) For a free particle moving in one dimension, write down the equation of motion for the position observable X in the Heisenberg picture. Solve this equation to obtain an expression for $X(t)$. Finally, write down an expression for $X(t)$ in the position-coordinate representation.

Hint: the Heisenberg equation of motion for an observable A with no explicit time dependence is

$$i\hbar \frac{d}{dt} A = [A, H]$$

Fall 2005 Prelims
Track II, Day 1
Question 5

In studying the emission of electrons from metals it is necessary to take into account the fact that electrons with energy sufficient to escape from the metal can, according to quantum mechanics, undergo reflection at the surface of the metal. Consider a one-dimensional model with the potential $V = -V_0$ for $x < 0$ (inside the metal and denoted as region 1) and $V = 0$ for $x > 0$ (outside the metal and called region 2). Assume the electron is inside the metal and has energy $E > 0$.

(a) 1 point: What is the form of the wavefunction in region 1?

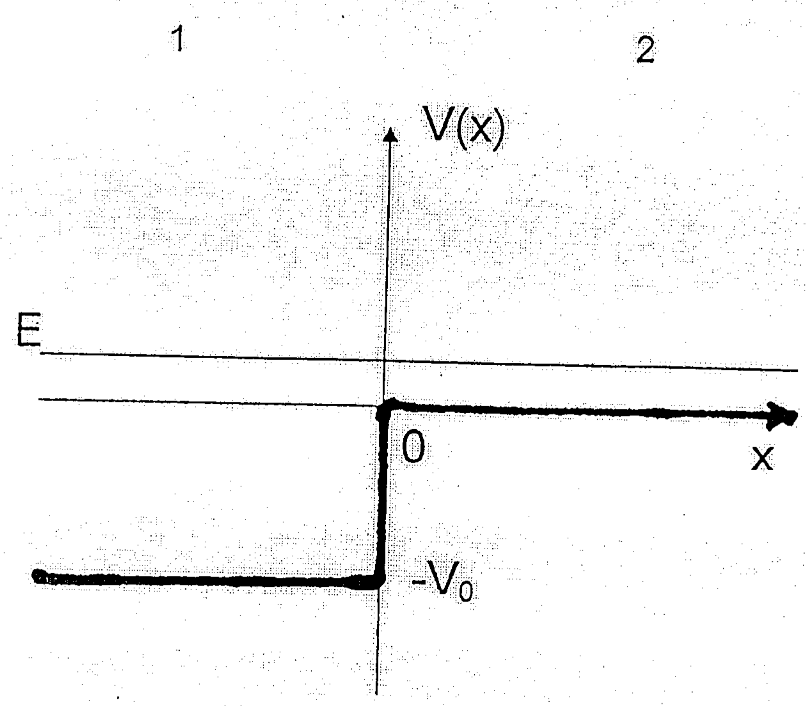
(b) 1 point: What is the form of the wavefunction in region 2?

(c) 3 points: What are the boundary conditions and what do they determine about the wavefunctions?

(d) 3 points: Derive an expression for the reflection coefficient R of an electron of energy $E > 0$ at the surface of the metal in terms of E and V_0.

(e) 1 point: What is R for $E << V_0$?

(f) 1 point: What is R for $E >> V_0$?



Fall 2005 Prelims
Track II, Day 1
Question 6

Quantized Field

(a) 1 point. Write down the interaction Hamiltonian between an atom and optical field in the dipole approximation.

(b) 1 point. Write down, in terms of the annihilation and creation operators a and $a^{\dagger}$, the single mode electromagnetic field Hamiltonian.

(c) 1 point. Find the corresponding eigenenergies of the field.

(d) 2 points. Sketch pictures of the energy levels of (i) an uncoupled two-level atom and (ii) a single mode quantized electromagnetic field.

(e) 3 points. Draw a picture of the coupled system energy levels for a two-level atom in resonance with a single mode quantized field according to the Jaynes-Cummings model.

(f) 2 points. Show how the transitions between the dressed states of a two-level atom resonantly and coherently driven by a cw single mode classical field leads to a three-peak spectrum.

TRACK II
DAY 2

WRITTEN PRELIM EXAM – SECOND DAY

Fall 2005

September 21, 2005
8:30 a.m. to 12:00 p.m.

Answer all six questions. Start each answer on a new page.

In the upper right hand corner of each sheet you hand in, put your code number (NO NAMES PLEASE) and the problem number. Staple together all sheets for a given problem.

Insert your answers into the manila envelope supplied and note which problems you have answered on the outside of the envelope.

The following are some helpful items:

$h = 6.625 \times 10^{-34}\ \mathrm{J \cdot s} = 4.134 \times 10^{-15}\ \mathrm{eV \cdot s}$

$e = 1.6 \times 10^{-19}\ \mathrm{C}$

$c = 3.0 \times 10^{8}\ \mathrm{m/s}$

$k_B = 1.38 \times 10^{-23}\ \mathrm{J/K}$

$\sigma = 5.67 \times 10^{-8}\ \mathrm{W/K^4 \cdot m^2}$

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}$

$\mu_0 = 1.26 \times 10^{-6}\ \mathrm{H/m}$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = 2\cos^2 A - 1$

$\cos 2A = 1 - 2\sin^2 A$

$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$

$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$\nabla\phi\psi = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\phi$

$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

$\nabla \cdot (\nabla \times \mathbf{F}) = 0$

$\nabla \times (\phi\mathbf{F}) = \phi(\nabla \times \mathbf{F}) + \nabla\phi \times \mathbf{F}$

$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

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$\oint_S (\mathbf{F} \cdot \mathbf{n})\, da = \int_V (\nabla \cdot \mathbf{F})\, d^3x$

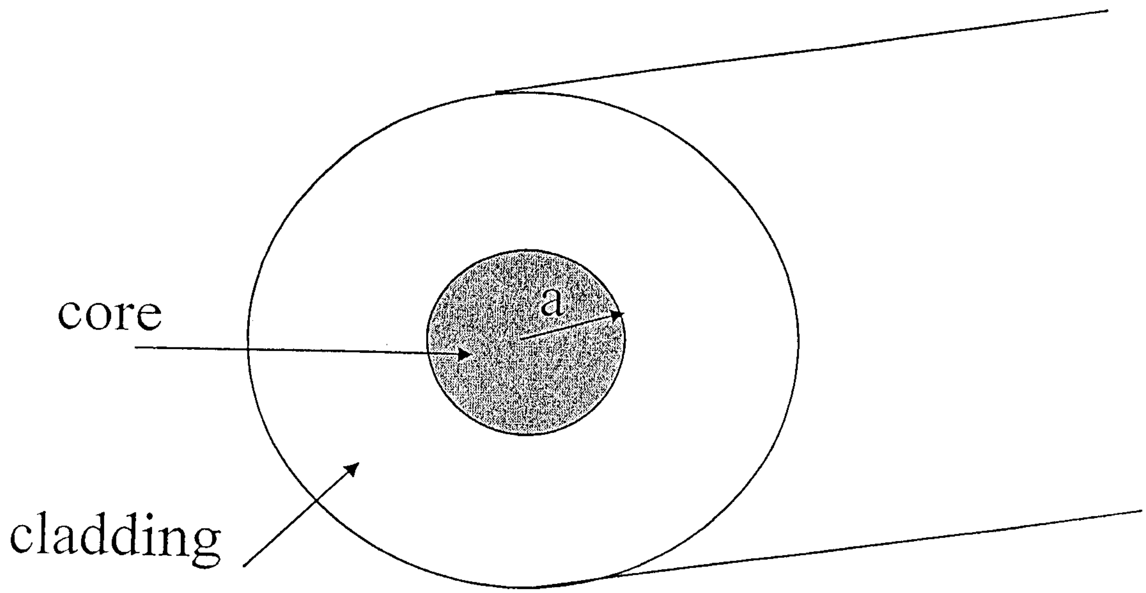
$\oint_C \mathbf{F} \cdot d\ell = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da$

$\oint_S \phi\mathbf{n}\, da = \int_V \nabla\phi\, d^3x$

$\oint_S \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_V [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

$\oint_S (\mathbf{n} \times \mathbf{F})\, da = \int_V (\nabla \times \mathbf{F})\, d^3x$

Fall 2005 Prelims
Track II, Day 2
Question 7



Consider light propagation in a step-index optical fiber (pictured above) with a core radius a, core refractive index n_{core} and cladding refractive index n_{clad}. The "characteristic equation" for a step index fiber can be written as:

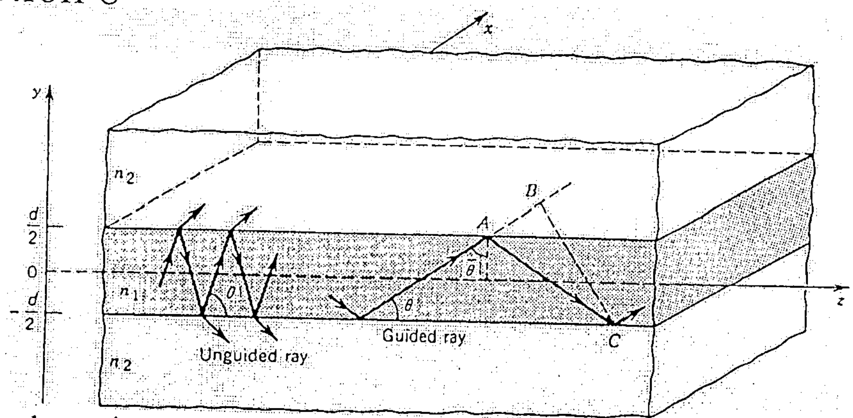
$$\frac{\beta^2 \nu^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = \left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

a) In analyzing step index optical fibers a *weakly guiding* approximation is often justified. Explain what it means and show how the "characteristic equation" simplifies to a form that does not explicitly contain β (no need to use Bessel function relations). **(4 points)**

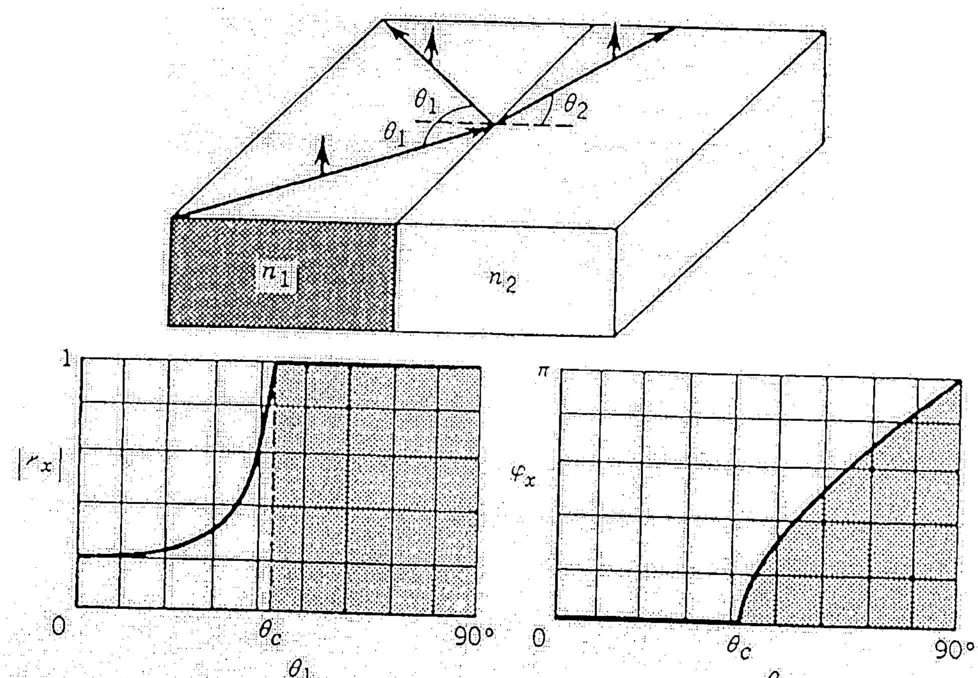
b) Explain the meaning of transverse modes in a step index fiber and show how the "characteristic equation" simplifies (no need to use Bessel function relations). **(4 points)**

c) With *weakly guiding* approximation, sketch the electric and magnetic field lines for the LP_{01} -mode in a cross section of a single mode fiber (indicate also the radial dependence of the fields). Briefly explain the degeneracy of the mode. **(2 points)**

Fall 2005 Prelims
Track II, Day 2
Question 8



The bouncing ray picture of guided waves in a planar slab waveguide.



The magnitude $|r_x|$ and phase ϕ_x of the reflection coefficient for internal reflection of a TE wave on a planar interface. The phase is given by the expression:

$$\tan\frac{\phi_x}{2} = \frac{\left(\sin^2\theta_1 - \sin^2\theta_c\right)^{1/2}}{\cos\theta_1}.$$

1. Find an implicit expression for the bouncing angles θ_m (see the bouncing ray picture above) for the guided waves in terms of the refractive indices n_1 and n_2, the core thickness d, and the vacuum wavelength λ_0. (The trig identity $\cos 2\theta = 1 - 2\sin^2\theta$ may be helpful.) **(7 points)**

2. Very briefly describe the origin of waveguide dispersion in the bouncing ray picture. **(3 points)**

Fall 2005 Prelims
Track II, Day 2
Question 9

(a) An ideal, thin, positive lens of focal length f converts an incident plane wave of wavelength l to a converging spherical wave that comes to a focus a distance f from the lens. Use this information to construct an expression for the amplitude transmittance of the lens in the Fresnel approximation.

(b) Consider a general monochromatic wave of wavelength λ incident on the lens. Denote the scalar wave amplitude of the wave in the plane of the lens as $u_0(x,y)$. Again using the Fresnel approximation, show that the wave amplitude in the back focal plane of the lens is related to the 2D Fourier transform of $u_0(x,y)$.

(c) As a specific illustration of the result of part (b), place a square aperture of side L in the plane of the lens and illuminate it with a monochromatic plane wave at normal incidence. Compute the *irradiance* in the back focal plane of the lens.

(d) Sketch the result of part (c), labeling the axes carefully and indicating key dimensions.

All parts carry equal weight.

Fall 2005 Prelims
Track II, Day 2
Question 10

In the angular spectrum view of optical propagation, the monochromatic (wavelength λ) complex amplitude in a plane is decomposed into plane-wave components. The propagation of these components is then effected by applying the transfer-function of free-space. The angular-spectrum components are then inverse Fourier transformed to yield the propagated complex amplitude. For propagation from the x-y plane to a parallel plane at $z_0 > 0$, the transfer function is

$$\exp\left[\frac{2\pi i z_0}{\lambda}\sqrt{1-\lambda^2\left(\xi^2+\eta^2\right)}\right] \text{ for } \xi^2+\eta^2 \le \frac{1}{\lambda^2} \text{ and } \exp\left[-\frac{2\pi z_0}{\lambda}\sqrt{\lambda^2\left(\xi^2+\eta^2\right)-1}\right] \text{ for } \xi^2+\eta^2 > \frac{1}{\lambda^2}.$$

The Gaus function is defined as $\text{Gaus}(x) = \exp\left(-\pi x^2\right)$; its Fourier transform is $\text{Gaus}(\xi) = \exp\left(-\pi\xi^2\right)$.

The Fourier transform of the "scaled" Gaus function is $\mathscr{F}\left[\exp\left(-\pi a^2 x^2\right)\right] = \dfrac{\exp\left(-\pi\dfrac{\xi^2}{a^2}\right)}{a}$ for any complex constant "a" as long as $\text{Re}\left(a^2\right) \ge 0$.

The complex-amplitude at $z = 0$, for this problem , is the Gaus function $\text{Gaus}\left(\sqrt{x^2+y^2}\right)$.

a. (3 Points)

What is the angular spectrum at the plane $z = 0$?

b. (3 Points)

What is the angular spectrum at the plane $z = z_0$?

c. (4 Points)

For small spatial frequencies, we sometimes use the Taylor series to approximate the transfer function by $\exp\left(\dfrac{2\pi i z}{\lambda}\right)\exp\left[-i\pi\lambda z\left(\xi^2+\eta^2\right)\right]$ for all ξ and η. In this approximation, propagate the complex amplitude from $z = 0$ to $z = z_0$. Perform the inverse Fourier transformation at the plane z_0 to find the complex amplitude in that plane. What is the diameter of the propagated wave's irradiance pattern? Remember that the root-mean-square full-width of the Gaussian is σ when the Gaussian is expressed as proportional to $\exp\left[\dfrac{-x^2}{2\sigma^2}\right]$.

Fall 2005 Prelims
Track II, Day 2
Question 11

Consider a dielectric medium (modeled by Lorentz oscillators) with the frequency-dependent dielectric function

$$\varepsilon(\omega) = \varepsilon_\infty \frac{\omega_L^2 - \omega^2}{\omega_T^2 - \omega^2}$$

where ω_L and ω_T (with $\omega_T < \omega_L$) are the longitudinal and transverse frequencies, respectively. Produce a qualitative sketch of the polariton dispersion $\omega(k)$, following the instructions given below.

Instructions:

First, specify the general polariton dispersion relation in a medium with arbitrary $\varepsilon(\omega)$. In other words, specify the generalization of the light dispersion in vacuum (here formulated for the squared quantities) $\frac{\omega^2}{c^2} = k^2$, where c is the light velocity in vacuum. (Assume the system to be infinitely large so that any surface or interface effects can be neglected.)

Secondly, using the dielectric function given above, discuss the dispersion $\omega(k)$ (with ω and k real) using the following five steps. (i) Show that the polariton dispersion has a gap (which you need to specify). (ii) Discuss the limit of $\omega(k) \to 0$ (i.e. $\omega(k) << \omega_L, \omega_T$). Use the LST relation $\frac{\omega_L^2}{\omega_T^2} = \frac{\varepsilon_0}{\varepsilon_\infty}$ with $\varepsilon_0 > \varepsilon_\infty$ to simplify your result. (iii) Discuss the limit of $\omega(k) \to \infty$ (i.e. $\omega(k) >> \omega_L, \omega_T$). (iv) Discuss the limit of $\omega(k) \to \omega_L$. (v) Discuss the limit of $\omega(k) \to \omega_T$.

Finally, sketch the dispersion relation, properly identifying the gap and labeling all four asymptotics.

(10 points)

Fall 2005 Prelims
Track II, Day 2
Question 12

(a) Consider the bandstructure of GaAs shown in Fig. 1. In the figure, indicate the lowest conduction band, the low-temperature chemical potential and the lowest-frequency direct optical transition (without electron-hole Coulomb interaction). In the lowest conduction band, which effective mass is larger, that at the Γ point or that at the X point?

(2 points)

(b) Consider the highest valence band of GaAs. The Δ line connecting Γ_8 and X_7 can be approximately written as

$$\varepsilon_k^\Delta = A\frac{1}{2}\left[\cos(\pi k / k_X) - 1\right]$$

State the general relationship between the velocity $\vec{v}_{\vec{k}}$ and the bandstructure $\varepsilon_{\vec{k}}$ and derive an analytic expression for the velocity v_k^Δ along Δ. Estimate the maximum speed $|v_{max}^\Delta|$ (in units of m/s) along Δ, assuming a lattice constant of $a = 5\text{Å}$ and using an estimate for the bandwidth of the Δ line based on the figure (indicate the energy interval in the figure). Note that the wave vector at X is $\vec{k}_X = \dfrac{2\pi}{a}(1,0,0)$ and use $\hbar = 6.58 \times 10^{-16}\, eVs$. Hint: if you don't remember the general relationship between the velocity and the bandstructure, recall the simple case of a parabolic band.

(8 points)

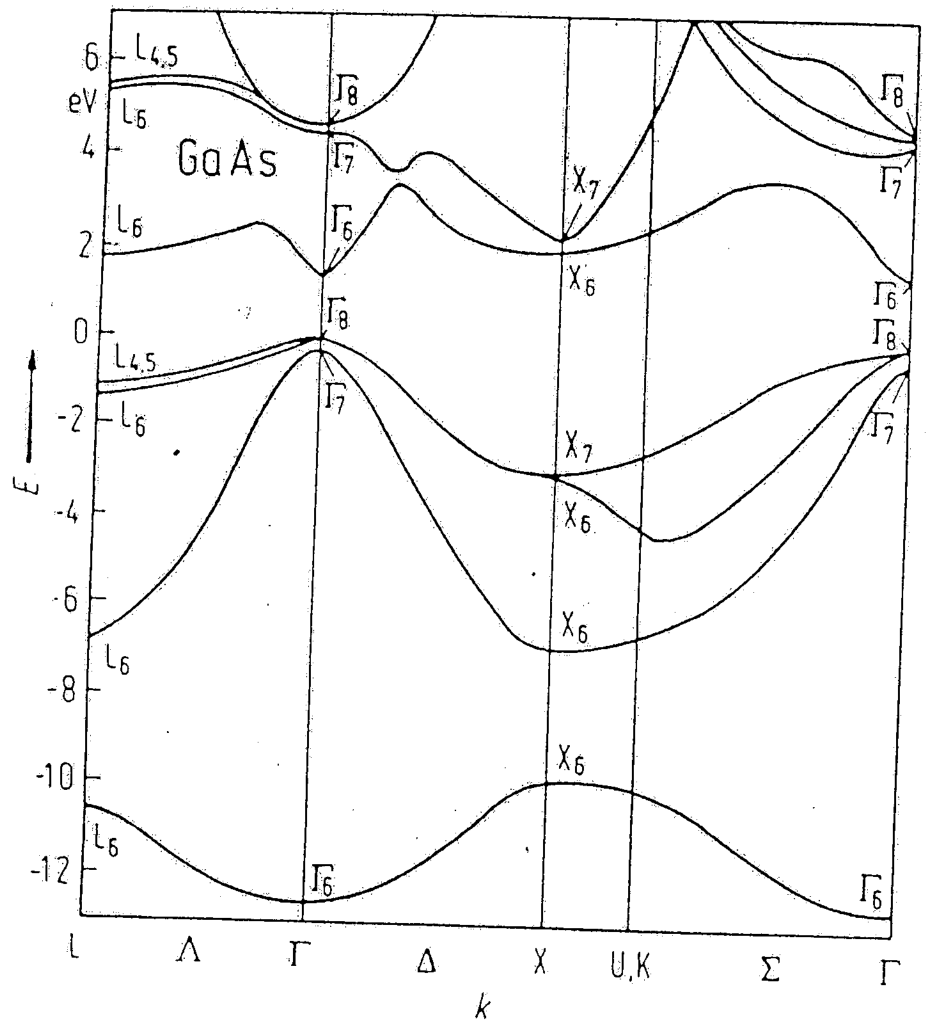


Fig. 1: Bandstructure (energy in units of eV) of GaAs

(Don't forget to write your name on the figure page and hand it in together with your solution.)