

Optical Sciences PhD Qualifying Exam:  
Instructions and Equation Sheet

Answer all four questions on the following pages. Start each answer on a new sheet of paper.

In the upper right hand corner of each sheet, write your code number (NO NAMES PLEASE) and the problem number (course number) that you are answering. Attach together all sheets for a given problem in order.

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The following are some helpful items:

$$h = 6.625 \times 10^{-34} \text{ J} \cdot \text{s} = 4.134 \times 10^{-15} \text{ eV} \cdot \text{s}$$

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$$c = 3.0 \times 10^8 \text{ m/s}$$

$$k_B = 1.38 \times 10^{-23} \text{ J/K}$$

$$\sigma = 5.67 \times 10^{-8} \text{ W/K}^4 \cdot \text{m}^2$$

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ F/m}$$

$$\mu_0 = 1.26 \times 10^{-6} \text{ H/m}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

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$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$$

$$\nabla\phi\psi = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$$

$$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$$

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$$

$$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\phi$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

$$\nabla \times (\phi\mathbf{F}) = \phi(\nabla \times \mathbf{F}) + \nabla\phi \times \mathbf{F}$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$$

$$\nabla \times \nabla\phi = 0$$

$$\oint_S (\mathbf{F} \cdot \mathbf{n}) da = \int_V (\nabla \cdot \mathbf{F}) d^3x$$

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} da$$

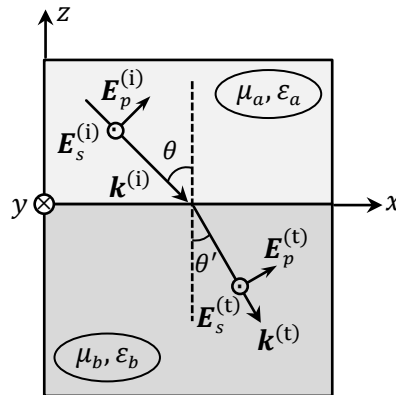
$$\oint_S \phi \mathbf{n} da = \int_V \nabla\phi d^3x$$

$$\oint_S \mathbf{F}(\mathbf{G} \cdot \mathbf{n}) da = \int_V [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}] d^3x$$

$$\oint_S (\mathbf{n} \times \mathbf{F}) da = \int_V (\nabla \times \mathbf{F}) d^3x$$

**System of units: SI (or MKSA)**

2) Consider the flat interface between two linear, isotropic, homogeneous media specified by their relative permeability and permittivity at the incidence frequency, namely,  $(\mu_a, \epsilon_a)$  for the medium above, and  $(\mu_b, \epsilon_b)$  for the medium below the interface. These material parameters (i.e.,  $\mu_a, \mu_b, \epsilon_a, \epsilon_b$ ) are assumed to be real-valued and positive. A homogeneous plane-wave of frequency  $\omega$  arrives at the interfacial  $xy$ -plane; the plane of incidence is  $xz$ , the incidence angle is  $\theta$ , and the  $E$ -field components of the incident beam are  $\mathbf{E}_p^{(i)}$  and  $\mathbf{E}_s^{(i)}$ , as indicated in the figure. The  $E$ -field components of the transmitted beam, also a homogeneous plane-wave, are  $\mathbf{E}_p^{(t)}$  and  $\mathbf{E}_s^{(t)}$ , and the angle between the transmitted  $k$ -vector and the surface-normal is  $\theta'$ , as shown.



- 2 pts a) Invoking the dispersion relation  $\mathbf{k} \cdot \mathbf{k} = (\omega/c)^2 \mu(\omega) \epsilon(\omega)$ , write expressions for the  $k$ -vectors of the incident and transmitted plane-waves shown in the above figure. ( $c = 1/\sqrt{\mu_0 \epsilon_0}$  is the speed of light in vacuum.)
  - 2 pts b) Invoking Maxwell's boundary conditions, explain why the transmitted wave has the same frequency  $\omega$  as the incident wave. What do these boundary conditions reveal about the relation between  $\theta$  and  $\theta'$ ?
  - 2 pts c) Use Maxwell's third equation,  $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$ , to determine both the incident and the transmitted  $H$ -field components. (As usual,  $\mathbf{B} = \mu_0 \mu(\omega) \mathbf{H}$ ; you may use the impedance of free space,  $Z_0 = \sqrt{\mu_0 / \epsilon_0}$ , to simplify the equations.)
  - 2 pts d) Find the conditions under which the reflected beam for the  $p$ -polarized incident light vanishes.
  - 2 pts e) Find the conditions under which the reflected beam for the  $s$ -polarized incident light vanishes.
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**Ph.D. Qualifying Exam, Fall 2022**

**Day 1**

**OPTI 502 Question**

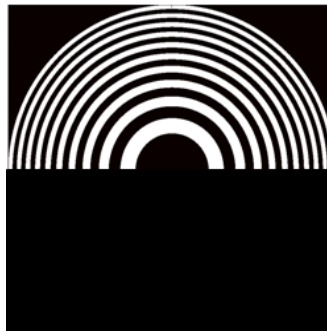
Consider a thin lens model of the lens in your mobile phone. Assume the focal length is 5 mm and the image sensor diagonal is 6 mm. The stop aperture is located at the thin lens, the lens F-number is 2.8, and there are no lens aberrations.

1. Determine what is the angular field of view in degrees required to fill the sensor.
2. Geometrically determine what the depth of focus is to maintain a spot size of 0.004 mm in diameter
3. For a wavelength of 0.0005 mm, determine what the diameter of the Airy disk is.
4. Determine what the lens angular resolving power is.
5. Determine the image distance from the lens nodal points when the object is located at 250 mm in front of the lens.
6. When the object is located at 250 mm from the lens and the object plane is tilted 5 degrees with respect to a line perpendicular to the lens optical axis, determine the image plane tilt. Make a NEAT drawing of the lens, the tilted object and the tilted image plane. Include the marginal and chief rays.
7. Determine the lens Lagrange invariant.

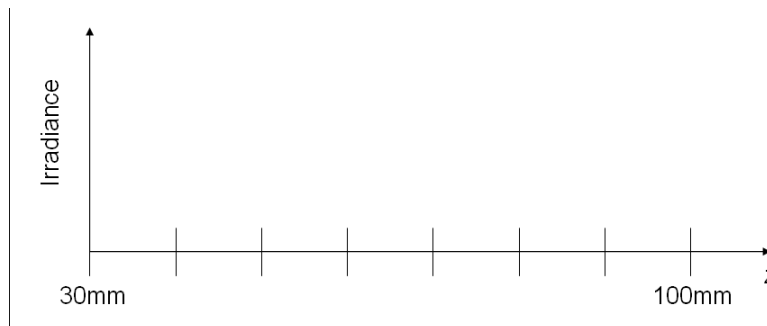
Question 6 is worth 4 points and all others are worth 1 point each.

**Ph.D. Qualifying Exam, Fall 2022**  
**Day 1**  
**OPTI 505R**

A simple binary amplitude (0 or 1 transmission) Fresnel zone plate (FZP) is fabricated as shown below, with one-half of the zone plate covered by an opaque mask. White areas indicate unity transmission. The plate and mask are illuminated by an on-axis plane wave with  $\lambda = 500\text{nm}$ . The zone plate is designed with 10 open zones ( $N_{\text{OFZ}} = 10$ ), and its primary focus (without the half-plane opaque mask) is designed for a primary focus at a distance of  $f_1 = 100\text{mm}$  behind the plate. The  $z$  optical axis is defined normal to the plate at the center of the rings. [Hint: Recall that the focal orders are located at the corresponding Fresnel numbers  $N_f$  of the first zone of the FZP.]



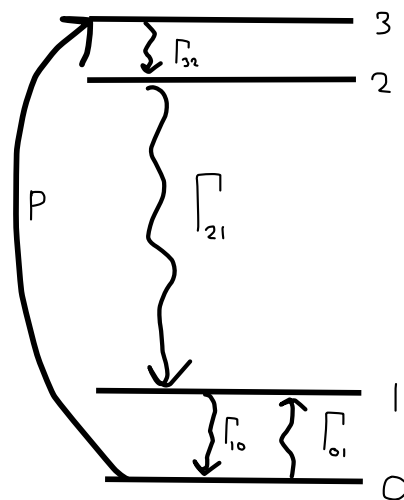
- 1.) (2pts) Calculate the focal distances of the second and third focal orders.
- 2.) (2 pts) How many times more intense is the on-axis irradiance at the  $f_1 = 100\text{mm}$  focal order than the on-axis irradiance of the plane wave illuminating the zone plate?
- 3.) (2 pts) How many times more intense is the on-axis irradiance at the second focal order than the on-axis irradiance of the plane wave illuminating the zone plate?
- 4.) (2 pts) How many times more intense is the on-axis irradiance at the third focal order than the on-axis irradiance of the plane wave illuminating the zone plate?
- 5.) (2pts) Replace the opaque half-plane mask with a phase plate that alters the phase of transmitted light relative to the uncovered section of the zone plate by  $\pi$  radians in transmission, but does not affect amplitude. On the graph below, sketch the on-axis irradiance from a distance of 30mm behind the plate to 100mm behind the plate. Indicate  $z$  positions of maxima and minima (if any). State any assumptions that you make.



Fall 2022 Qualifying Exam  
 Day 1  
 OPTI 511R / 570 / 544  
 (10 points total)

### Optical gain in a multi-level system

Consider a 10 cm long homogeneously broadened gain medium. The relevant energy levels for this system are shown in the figure, with population densities  $N_0, N_1, N_2, N_3$ . The total population density of this closed system is  $N_T = 1 \times 10^{19} \text{cm}^{-3}$ . The population transition rates ( $\Gamma_{ij}$ ) are also shown, where  $\Gamma_{21}$  is due entirely to spontaneous emission. Note that between levels 1 and 0 there is a population changing process that can drive transitions from  $1 \rightarrow 0$  and from  $0 \rightarrow 1$  (this may be the case for close lying levels in a molecule for example, where the energy difference is small compared to the thermal energy). An external pumping rate  $P$  is required for population inversion. The maximum gain cross section for the  $2 \rightarrow 1$  transition is  $\sigma(\nu_o) = 3 \times 10^{-21} \text{cm}^2$ , and the on-resonance saturation intensity is  $I_{sat}(\nu_o) = 100 \text{W/cm}^2$ . **In this problem you can assume that  $\Gamma_{32}$  is infinite (fast decay such that  $N_3 \approx 0$  in the steady state) and that  $\Gamma_{10} = \Gamma_{01}$ .**



- (a) [2 points] Write down the rate equations for levels  $N_2, N_1$  and  $N_0$  given a pumping rate  $P$ .
- (b) [3 points] Using results from part (a), solve for  $N_2/N_1$  in the steady-state, expressing your answer in terms of  $P, \Gamma_{21}$ , and  $\Gamma_{10}$ .
- (c) [3 points] Assuming that  $\Gamma_{10} \gg P$  and a pumping rate  $P = 2\Gamma_{21}$ , calculate the on resonance small-signal gain coefficient for monochromatic light resonant with the  $2 \rightarrow 1$  transition .
- (d) [2 points] For monochromatic light near the atomic resonance ( $\nu \approx \nu_0$ ), calculate the intensity at which the gain coefficient is reduced to half of its small-signal value.

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$$\oint_S (\mathbf{n} \times \mathbf{F}) da = \int_V (\nabla \times \mathbf{F}) d^3x$$

## Opti 501

System of units: SI (or MKSA)

1) An electromagnetic plane-wave of frequency  $\omega$  propagates along the unit-vector  $\hat{\mathbf{k}}$  in free space. The plane-wave's  $E$ -field is specified as  $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ . Assume that the  $k$ -vector is real-valued (i.e.,  $\hat{\mathbf{k}}$  is real), but that the field amplitude  $\mathbf{E}_0$  is complex; that is,  $\mathbf{E}_0 = \mathbf{E}'_0 + i\mathbf{E}''_0$ , where  $\mathbf{E}'_0$  and  $\mathbf{E}''_0$  are a pair of real-valued (but otherwise arbitrary) vectors.

- 1 pt a) Write the expression that relates  $\mathbf{k}$  to  $\omega$ ,  $c$  (the speed of light in vacuum), and  $\hat{\mathbf{k}}$ .
- 1 pt b) Invoke Maxwell's first equation,  $\nabla \cdot \mathbf{D} = \rho_{\text{free}}$ , to show that  $\hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$ . Explain why this relation implies that both  $\mathbf{E}'_0$  and  $\mathbf{E}''_0$  are perpendicular to the unit-vector  $\hat{\mathbf{k}}$ .
- 2 pts c) Explain how an arbitrary pair of  $\mathbf{E}'_0$  and  $\mathbf{E}''_0$  could give rise to an elliptical state of polarization of the plane-wave. Under what circumstances will the plane-wave be linearly polarized? Under what circumstances will the plane-wave be right (or left) circularly polarized?
- 2 pts d) Use Maxwell's third equation,  $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$ , to find an expression for the plane-wave's  $H$ -field in terms of  $\omega$ ,  $\hat{\mathbf{k}}$ ,  $\mathbf{E}_0$ , and the impedance  $Z_0$  of free space.
- 2 pts e) Derive expressions for the time-averaged  $E$ -field energy-density  $\langle \mathcal{E}_E(\mathbf{r}, t) \rangle$ , time-averaged  $H$ -field energy-density  $\langle \mathcal{E}_H(\mathbf{r}, t) \rangle$ , and time-averaged Poynting vector  $\langle \mathbf{S}(\mathbf{r}, t) \rangle$  in terms of the parameters  $\epsilon_0, \mu_0, \hat{\mathbf{k}}, \mathbf{E}'_0, \mathbf{E}''_0$ . (Recall that  $\mathcal{E}_E, \mathcal{E}_H$ , and  $\mathbf{S}$  must be real-valued.)
- 2 pts f) Consider the linear superposition of two plane-waves of the same frequency  $\omega$ , propagating in the same direction  $\hat{\mathbf{k}}$ , but having different polarization states  $\mathbf{E}_{01}$  and  $\mathbf{E}_{02}$ . Using an arbitrary pair of complex coefficients  $(\alpha, \beta)$ , the superposed  $E$ -field may be written as

$$\mathbf{E}(\mathbf{r}, t) = (\alpha \mathbf{E}_{01} + \beta \mathbf{E}_{02}) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}.$$

The two polarization states are said to be mutually orthogonal if, for all values of  $\alpha$  and  $\beta$ , the energy-density of the superposition equals the corresponding linear combination of the individual energy densities of the two plane-waves. (This property also extends to the Poynting vector.) Under what circumstances can  $\mathbf{E}_{01}$  and  $\mathbf{E}_{02}$  be considered to be mutually orthogonal?

**Hint:**  $\sin(2x) = 2 \sin x \cos x$ ;  $\cos(2x) = \cos^2 x - \sin^2 x$ ;  $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ ;  
 $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ ;  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ ;  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ .

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## Ph.D. Qualifying Exam, Fall 2022

### Day 2

#### OPTI 502 Question

Design an infinity-corrected (e.g., infinite-conjugate) object-space telecentric, non coverslip-corrected, microscope to fully image a 5 mm x 5 mm sample area with no vignetting onto a 25 mm x 25 mm sensor with 10 micron square pixels. The microscope should have an object-space diffraction limited resolution (e.g. minimum object separation) of exactly 1 micron according to the Rayleigh criterion at an operating wavelength of 655 nm and working distance of 4 mm. You may treat the objective and tube lenses as thin lenses that are aberration free. The microscope is imaging with no immersion medium ( $n_0=1$ ). The spacing between the two lenses is 54 mm.

- a) Provide focal lengths for both the objective and tube lens as well as object distance and image distance. (2 pt)
- b) Provide the stop location and diameter, diameters of the objective and tube lens, and entrance / exit pupil locations and size of the exit pupil. For large NA ( $NA > 0.25$ ), do not use the small angle approximation. (3 pt)
- c) Illustrate the system on a diagram with all distances and sizes (for object, image pupils, and lens separations). Show the chief and marginal ray. (1 pt)
- d) What are the first order effects of adding a 0.3 mm thick glass ( $n=1.5$ ) coverslip between the objective and sample? (0.5 pt)
- e) What is the depth of focus (as defined by geometrical optics - you may use small angle approximation for NA) in object space of the system assuming a maximum blur of one pixel size? (0.5 pt)
- f) What is the effect on resolution (as defined by the Rayleigh Criterion), of immersing the sample in oil ( $n_0=2.0$ ). (0.5 pt)
- g) Briefly explain (e.g., 1-2 sentences) in terms of marginal and chief rays how you would design a specular illumination system to illuminate the sample from behind? Would a source with spatial uniformity or angular uniformity be advantageous for having the most uniform illumination at the sample? (2.5pt)



For parts (a–c), a reflection grating with a rectangular surface is illuminated by a uniform plane wave with mean wavelength  $\lambda = 1 \mu\text{m}$  at an angle of incidence  $\theta_{\text{SRC}} = -30^\circ$  (Figure 1). The period of the 50% duty cycle grating is  $d = 10 \mu\text{m}$ , and the depth of the grating is  $h = 0.1 \mu\text{m}$ . The incident medium is vacuum. For parts (a–c), the answers should be reduced to numerical values.

- For the grating, what are the angles (in degrees) of the two diffracted orders  $\theta_m$  for  $m = -1, -2$ , as shown? (2 points)
- What is the absolute diffraction efficiency into the  $m = 0, -1, -2$  orders? You may assume that the grating is 100% reflective at all angles of incidence. (4 points)
- If the grating is  $L = 10 \text{ cm}$  wide in the  $y_s$ -direction, what is the approximate minimum spectral resolution  $\Delta\lambda_{\text{RES}}$  in the  $m = -1$  order? (Answers within an order of magnitude will be given full credit since we have not precisely defined a resolution criterion here.) (1 point)

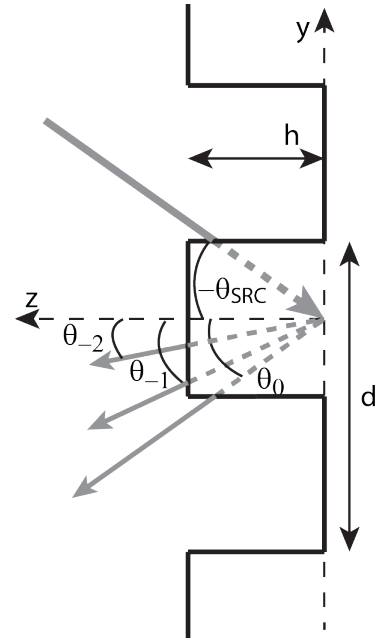


Figure 1: Rectangular reflection grating

For parts (d) and (e), consider the grating in Figure 2 with a sawtooth amplitude profile, assuming the same geometric parameters as the rectangular grating above.

- For the sawtooth grating, are the angles of the diffracted orders less than, greater than, or the same as for the rectangular grating? (1 point)
- Optimizing over all possible angles of incidence,  $\theta_{\text{SRC}}$ , and all values of  $h$ , which of the two types of gratings produce a higher diffraction efficiency in the  $m = -1$  order? What terminology is used to describe gratings designed to produce this effect? It is not necessary to do any calculations for this part. (2 points)

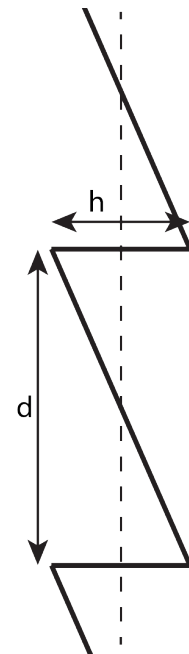


Figure 2: Sawtooth reflection grating

**Reference equations are included on the next page...**

Reference information:

The Fourier series coefficients for a grating of period  $d$  with a complex amplitude transmittance  $f_d(y_s)$  are given by:

$$c_m = \frac{1}{d} \int_{-\frac{d}{2}}^{\frac{d}{2}} f_d(y_s) e^{-\frac{j2\pi m y_s}{d}} dy_s = \mathcal{F}_m \left\{ f_d(y_s) \frac{1}{d} \text{rect} \left( \frac{y_s}{d} \right) \right\}$$

A selection of diffraction integrals:

$$U(x_0, y_0, z_0) = -j \frac{e^{jkz_0} e^{\frac{jk}{2z_0}(x_0^2 + y_0^2)}}{\lambda z_0} \iint U_S^+(x_s, y_s) \exp \left[ \frac{jk}{2z_0}(x_s^2 + y_s^2) \right] \exp \left[ -j \frac{2\pi}{\lambda z_0}(x_s x_0 + y_s y_0) \right] dx_s dy_s$$

$$U(x_0, y_0, z_0) = \iint \left[ \iint U_S^+(x_s, y_s) e^{-j2\pi(x_s \xi + y_s \eta)} dx_s dy_s \right] \exp(jk\gamma z_0) e^{j2\pi(x_0 \xi + y_0 \eta)} d\xi d\eta,$$

where  $\gamma = \sqrt{1 - (\lambda\xi)^2 - (\lambda\eta)^2}$  is the  $z$ -direction cosine.

$$U(x_0, y_0, z_0) = -j \frac{e^{jkz_0} e^{\frac{jk}{2z_0}(x_0^2 + y_0^2)}}{\lambda z_0} \iint U_S^+(x_s, y_s) \exp \left[ -j \frac{2\pi}{\lambda z_0}(x_s x_0 + y_s y_0) \right] dx_s dy_s$$

Fourier pairs:

$f(x)$	$g(\xi) \equiv \int_{-\infty}^{\infty} f(x) e^{-j2\pi\xi x} dx$
1	$\delta(\xi)$
$f(x)e^{jax}$	$g\left(\xi - \frac{a}{2\pi}\right)$
$\sin(ax)$	$\frac{\delta\left(\xi - \frac{a}{2\pi}\right) + \delta\left(\xi + \frac{a}{2\pi}\right)}{2j}$
$\text{rect}_A(x) \equiv \begin{cases} 0 & \text{for }  x  > A/2 \\ 1/2 & \text{for }  x  = A/2 \\ 1 & \text{for }  x  < A/2 \end{cases}$	$A \text{sinc}(A\xi) \equiv A \frac{\sin(\pi A\xi)}{\pi A\xi}$
$\text{comb}_T(x) \equiv \sum_{m=-\infty}^{\infty} \delta(x - mT)$	$\frac{1}{T} \text{comb}_{\frac{1}{T}}(\xi)$

Consider a particle of mass  $m$  confined to a 1D harmonic potential  $V(x)$  centered at  $x = 0$ . The potential is characterized by an oscillation frequency  $\omega$ .

1. Write down the Hamiltonian  $\hat{H}(\hat{x}, \hat{p})$  of this system in operator form. (1 pt)

2. What are the energies of the ground state  $|0\rangle$  and first excited state  $|1\rangle$ ? (1 pt)

3. Sketch the wavefunction of  $|0\rangle$  and  $|1\rangle$  in position space. (2 pt)

4. Suppose that at time  $t = 0$ , the particle is in an initial state

$$|\psi(0)\rangle = c_0|0\rangle + c_1|1\rangle \quad (1)$$

where  $c_0$  and  $c_1$  are real and positive and  $c_1 \ll c_0 \approx 1$ . Argue graphically or show explicitly that, in position space, the wavefunction of  $|\psi(0)\rangle$  looks like that of a slightly displaced ground state. (2 pt)

5. Without derivation, give an expression for the state vector  $|\psi(t)\rangle$  at a later time  $t$  in terms of  $|0\rangle$ ,  $|1\rangle$  and  $\omega$ . (1 pt)

6. Derive an expression for the time dependent expectation value  $\langle \hat{x} \rangle(t)$  in terms of  $c_0$ ,  $c_1$ ,  $\omega$ , and the spatial overlap integral  $\langle 0|\hat{x}|1\rangle$ . (2 pt)

7. What features does  $\langle \hat{x} \rangle(t)$  have in common with the classical trajectory  $x(t)$  of a harmonic oscillator? (1 pt)