WRITTEN PRELIM EXAM – FIRST DAY

Fall 2009

September 22, 2009
8:30 a.m. to 12:00 p.m.

Answer all six questions. Start each answer on a new page.

In the upper right hand corner of each sheet you hand in, put your code number (NO NAMES PLEASE) and the problem number. Staple together all sheets for a given problem.

Insert your answers into the manila envelope supplied and note which problems you have answered on the outside of the envelope.

The following are some helpful items:

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$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

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$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$\nabla\phi\psi = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

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$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\phi$

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$\oint_S (\mathbf{F} \cdot \mathbf{n})\, da = \int_V (\nabla \cdot \mathbf{F})\, d^3x$

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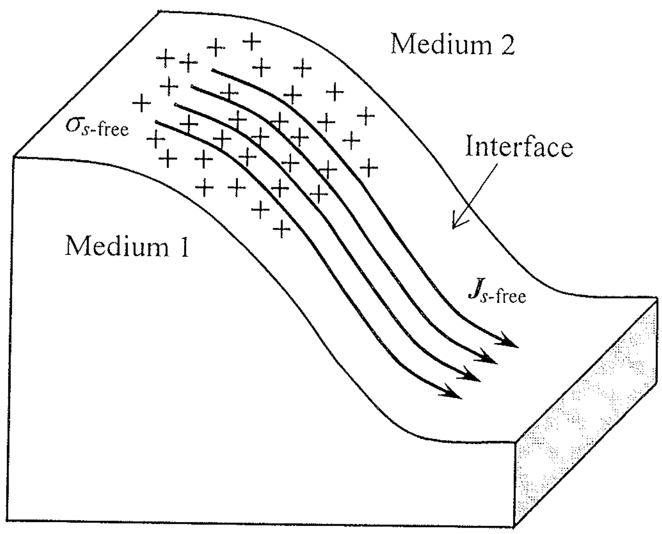
$\oint_S \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_V [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

$\oint_S (\mathbf{n} \times \mathbf{F})\, da = \int_V (\nabla \times \mathbf{F})\, d^3x$

System of units: MKSA

Corresponding to the four Maxwell equations are four boundary conditions that relate the field components $\boldsymbol{E}_{\|}$, $\boldsymbol{H}_{\|}$, $\boldsymbol{D}_{\perp}$ and $\boldsymbol{B}_{\perp}$ on the two sides of a sharply defined interface between two neighboring media. The subscripts $\|$ and $\perp$ identify the local field components parallel and perpendicular to the interface, respectively. In general, the interface may contain a surface-charge-density $\sigma_{s\text{-free}}(\boldsymbol{r},t)$, and be host to a surface-current-density $\boldsymbol{J}_{s\text{-free}}(\boldsymbol{r},t)$. Here $\boldsymbol{r}=(x,y,z)$ is an arbitrary point at the interface, and t is an arbitrary instant of time. In what follows, $\boldsymbol{r}^+$ will be a point immediately above the interface, while $\boldsymbol{r}^-$ will be the corresponding point immediately below the interface.



(2 pts) a) Use Maxwell's first equation, $\nabla\cdot\boldsymbol{D}=\rho_{\text{free}}$, to relate $\boldsymbol{D}_{\perp}(\boldsymbol{r}^+,t)$ and $\boldsymbol{D}_{\perp}(\boldsymbol{r}^-,t)$.

(4 pts) b) Use Maxwell's second equation, $\nabla\times\boldsymbol{H}=\boldsymbol{J}_{\text{free}}+\partial\boldsymbol{D}/\partial t$, to relate $\boldsymbol{H}_{\|}(\boldsymbol{r}^+,t)$ and $\boldsymbol{H}_{\|}(\boldsymbol{r}^-,t)$.

(2 pts) c) Use Maxwell's third equation, $\nabla\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t$, to relate $\boldsymbol{E}_{\|}(\boldsymbol{r}^+,t)$ and $\boldsymbol{E}_{\|}(\boldsymbol{r}^-,t)$.

(2 pts) d) Use Maxwell's fourth equation, $\nabla\cdot\boldsymbol{B}=0$, to relate $\boldsymbol{B}_{\perp}(\boldsymbol{r}^+,t)$ and $\boldsymbol{B}_{\perp}(\boldsymbol{r}^-,t)$.

A Petzval objective is to be used in conjunction with a cube beamsplitter. What is the size of the largest beamsplitter that will fit between the rear element of the Petzval objective and the image plane? Consider that the image is formed on the rear surface of the beamsplitter, and the beamsplitter is in contact with the rear element of the objective.

Details:

Cube Beamsplitter:	Index of refraction = 1.5
Petzval Objective:	Two 50 mm focal length thin lenses (in air) separated by 25 mm
Object:	The object is 100 mm to the left of the front element of the Petzval objective.

a. (2 pts) What wavelength of light (in nanometers) is needed to excite the hydrogen atom from its ground state ($E_1 = -13.6\text{eV}$) to one of its excited states with principal quantum number n=3?

b. (3 pts) Assume the hydrogen atom is initially in its ground state. Circularly polarized light propagating along the z-direction and at the wavelength found in part (a) is turned on at $t = 0$ and shut off at $t = t_1$. This light polarization carries positive angular momentum about the positive z axis. When the field is turned off, the light-atom coupling leaves the atom in a state such that it has a 50% chance of being found in its ground state. Write a normalized expression for the *time dependent* atomic wavefunction in terms of the hydrogen atom energy eigenstates ψ_{nlm} and $E_1$for times $t > t_1$. (For this part of the problem, ignore the effects of spontaneous emission and spin, and assume we are working within the 2-level atom approximation)

c. (2 pts) With the atom initially in the superposition state described in part (b), a measurement determines that the atom is in the quantum state characterized by $n = 3$. If a measurement is then made of the atom's orbital angular momentum projected along the z direction, what value will be measured?

d. (1 pts) Assume the atom is in the energy eigenstate state found in part (c). Taking into account spontaneous emission at optical frequencies, what are the possible states into which the atom can decay?

e. (2 pts) Using the following expression for the Einstein A coefficient and the appropriate matrix elements of hydrogen, determine the lifetime of this state (provide a numeric answer). Recall that if there are multiple decay channels from a given state, the total A coefficient for that state is given by $A_{total} = A_1 + A_2 + A_3...$

$A = \frac{\omega_0^3|\vec{\wp}|^2}{3\pi\epsilon_0\hbar c^3} = \left|\frac{\delta E}{E_1}\right|^3 \times \frac{|\vec{\wp}|^2}{e^2 a_0^2} \times 2.35 \times 10^9\ \text{s}^{-1}$, where δE is the energy difference between 2 states.

$$\langle l = 0,\, m = 0\, |\hat{\mathbf{r}}|\, l = 1,\, m = 0\rangle = (0,\, 0,\, \sqrt{\tfrac{1}{3}})$$
$$\langle l = 0,\, m = 0\, |\hat{\mathbf{r}}|\, l = 1,\, m = 1\rangle = (-\sqrt{\tfrac{1}{6}},\, -i\sqrt{\tfrac{1}{6}},\, 0)$$
$$\langle l = 0,\, m = 0\, |\hat{\mathbf{r}}|\, l = 1,\, m = -1\rangle = (\sqrt{\tfrac{1}{6}},\, -i\sqrt{\tfrac{1}{6}},\, 0)$$

$$\langle n = 1,\, l = 0\, |r|\, n = 2,\, l = 1\rangle = 1.29a_0$$
$$\langle n = 1,\, l = 0\, |r|\, n = 3,\, l = 1\rangle = 0.517a_0$$
$$\langle n = 2,\, l = 0\, |r|\, n = 3,\, l = 1\rangle = 3.07a_0$$
$$\langle n = 2,\, l = 1\, |r|\, n = 3,\, l = 0\rangle = 0.95a_0$$

The so-called coherent states of the electromagnetic field are described by quantum states that, in the Heisenberg picture, have the form

$$|\alpha\rangle = e^{-\alpha^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

(a) Write down the defining eigenvalue equation for the coherent state $|\alpha\rangle$.

(b) Explain in words why coherent states are of special importance in quantum optics.

(c) In the Heisenberg picture the creation and annihilation operators are time dependent. Write down expressions for $a(t)$, $a^+(t)$.

(d) Calculate expectation values of $|\alpha\rangle$ for the quadrature operators $\hat{X}(t) = [a(t) + a^+(t)]/2$, $\hat{Y}(t) = i[a(t) - a^+(t)]/2$. Compare to your answer in (b).

A coherent states $|\alpha\rangle_1$ enters through port 1, and the vacuum state $|0\rangle_2$ through port 2, of a beam splitter with transfer matrix

$$M = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

(e) Recalling that $|\alpha\rangle = \exp[\alpha a^+ - \alpha^* a]|0\rangle$, find the quantum state for the field in port 3. What does this result imply for the robustness of coherent states?

According to the Huygens-Fresnel principle and the Rayleigh-Sommerfeld diffraction integral, for a given initial field $U_S(r_s)$ on a plane at z =0, the field following propagation to an observation plane over a distance $z_0 \gg \lambda$ in vacuum may be expressed by as the integral

$$U_0(r_0) = \frac{1}{i\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_z \frac{e^{ikr_{0S}}}{r_{0S}} U_S(r_s)\, dx_s dy_s \ , \tag{1}$$

where $r_{0S} = \sqrt{(x_0 - x_S)^2 + (y_0 - y_S)^2 + z_0^2}$ is distance between point r_s on the diffracting plane and point r_0 on the observation plane, $k = 2\pi/\lambda$ and $\gamma_z = \frac{z_0}{r_{0S}}$.

a) (3 pts) Describe any key ideas and approximations that underlie reduction of the diffraction integral in Eq. (1) to its form in the Fresnel approximation

$$U_0(r_0) = \frac{e^{ikz_0}}{i\lambda z_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{ik}{2z_0}\left[(x_0-x_S)^2+(y_0-y_S)^2\right]} U_S(r_s)\, dx_s dy_s \ . \tag{2}$$

b) (2 pts) Next, describe approximations that underlie reduction of the Fresnel approximations in Eq. (2) to its form in the Fraunhofer region

$$U_0(r_0) = \frac{e^{ikz_0}}{i\lambda z_0} e^{\frac{ik}{2z_0}(x_0{}^2+y_0{}^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{ik}{z_0}(x_0x_S+y_0y_S)} U_S(r_s)\, dx_s dy_s \tag{3}$$

c) (2 pts) Hereafter consider an incident Gaussian at the waist $U_S(r_s) = F(0)e^{-r_s^2/\omega_0^2} = F(0)\ \mathrm{gaus}\left(\frac{x_s}{\omega_0\sqrt{\pi}}\right)\ \mathrm{gaus}\left(\frac{y_s}{\omega_0\sqrt{\pi}}\right)$, where $r_s = \sqrt{x_s^2 + y_s^2}$, with on-axis amplitude $F(0)$ and Gaussian spot size ω_0 that specifies spatial extent of the field. Based on your analysis from part (b), show that the Fraunhofer region arises for $z_0 \gg z_R$, where the Rayleigh range is given by $z_R = \pi\omega_0^2/\lambda$.

d) (3 pts) A simple circularly symmetric Gaussian beam may be written as

$$U(r) = F(z)e^{\frac{ikr^2}{2R(z)}}e^{-\frac{r^2}{\omega^2(z)}} \ ,$$

where $r = \sqrt{x^2 + y^2}$. Specializing to the Fraunhofer region and for the initial collimated Gaussian in part (c), find an expression for the radius of curvature $R(z)$ of the propagating

Gaussian, and show that Gaussian spot size evolves according to $\omega(z) = \omega_0\left(\frac{z}{z_R}\right)$. You might need the integral

$$\int_{-\infty}^{\infty} ds\, e^{-isq-as^2} = \sqrt{\frac{\pi}{a}}\, e^{-q^2/4a}$$

or the fact that

$F_\xi\left[\text{gaus}(x/a)\right] = a\,\text{gaus}(a\xi)$, where $F_\xi[\;]$ indicates Fourier transform.

According to the Huygens-Fresnel principle, for a given initial field $E(x', y', 0)$ at $z = 0$, the field following propagation over a distance $z > 0$ in vacuum may be expressed as the diffraction integral

$$E(x, y, z) = \frac{1}{i\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx' dy' E(x', y', 0) \left(\frac{e^{ikr}}{r} \right) \left(\frac{z}{r} \right), \qquad (1)$$

where $k = 2\pi/\lambda$, and $r = \sqrt{(x - x')^2 + (y - y')^2 + z^2}$.

(a - 3pts) Describe the key ideas and approximations that underly the reduction of the diffraction integral in Eq. (1) to its form in the Fresnel approximation

$$E(x, y, z) \approx \frac{e^{ikz}}{i\lambda z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx' dy' E(x', y', 0) e^{\frac{ik[(x-x')^2 + (y-y')^2]}{2z}}. \qquad (2)$$

(b - 2pts) Next describe the approximations that underly the reduction of the Fresnel approximation in Eq. (2) to its form in the Fraunhofer region

$$E(x, y, z) \approx \frac{e^{ikz}}{i\lambda z} e^{\frac{ik(x^2+y^2)}{2z}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx' dy' E(x', y', 0) e^{-\frac{ik(xx' + yy')}{z}}. \qquad (3)$$

(c - 2pts) Hereafter consider a Gaussian $E(x', y', 0) = F(0)e^{-(x'^2 + y'^2)/w_0^2}$ at its waist with on-axis amplitude $F(0)$, and Gaussian spot size w_0 that specifies the spatial extent of the incident field. Based on your analysis from part (b), show that the Fraunhofer region arises for $z >> z_0$ where the Rayleigh range is given by $z_0 = kw_0^2/2$.

(d - 3pts) A circular cross-section Gaussian beam may be written as

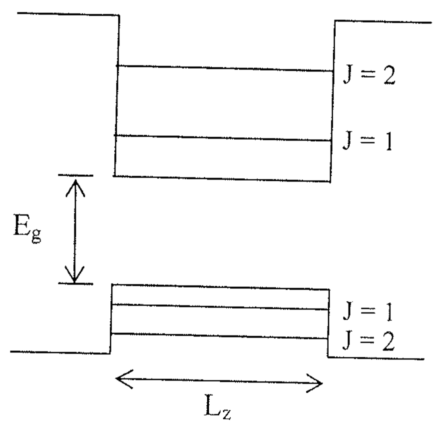
$$E(x, y, z) = F(z) e^{\frac{ik(x^2+y^2)}{2R(z)}} e^{-\frac{(x^2+y^2)}{w^2(z)}}.$$

Specializing to the Fraunhofer region, and for the initial collimated Gaussian beam above, find an expression for the radius of curvature $R(z)$ of the propagating Gaussian, and show the Gaussian spot size evolves according to $w(z) = w_0 \left(\frac{z}{z_0} \right)$. (The tabulated integral

$$\int_{-\infty}^{\infty} dx' e^{-ix'q - ax'^2} = \sqrt{\frac{\pi}{a}} e^{-q^2/4a},$$

will be needed.)

1. Consider a two-dimensional semiconductor with quantized valence and conduction band states as shown below.



(a) 1 point – Assuming infinite barrier heights, what are the allowed optical transitions between valence and conduction band quantized states?

(b) 3 points – Calculate transition energies for the following parameters in units of eV (again assuming infinite barrier heights):

$L_z = 50$ Å, $E_g = 1.5$ eV, $m_e = 0.1\ m_o$, $m_h = 10\ m_o$, m_o = electron mass = $9.1 \cdot 10^{-31}$ kg

$\hbar = 1.05 \cdot 10^{-34}$ joules - sec, 1eV = 1.602×10^{-19} J

(c) 2 points – Consider the optical transition from the J=1 valence band to the J=3 conduction band (not explicitly shown in the above figure). Is this transition allowed if the barrier heights are infinite? Could the answer be different if the barrier heights are finite (but large enough to ensure the states to be localized in the well)?

(d) 2 points – Plot the joint density of states for this semiconductor.

(e) 2 points – Plot the absorption spectrum by considering coulomb interaction between electrons and holes in this semiconductor.

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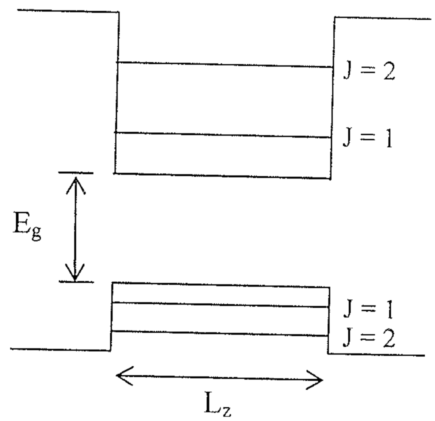

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(c) 2 points – What are the allowed optical transitions between the quantized valence and conduction band states when a large electric field is applied?

(d) 2 points – Plot the joint density of states for this semiconductor.

(e) 2 points – Plot the absorption spectrum by considering coulomb interaction between electrons and holes in this semiconductor.

Silicon photodiode

The five parts below are equally weighted, but taken together they should comprise a succinct but coherent description of what a silicon photodiode is and how it works.

a. What is meant by P-type and N-type silicon? Give an example of a dopant that will produce each type.

b. What is a PN junction? Why does it function as a diode?

c. Sketch a silicon photodiode, indicating the P-region, the N-region, the depletion region and the electrical contacts. Show the light incident on the device and indicate where it should be absorbed.

d. Sketch the current-voltage curve of the photodiode, in the dark and at two or three different levels of irradiance. Comment briefly on these curves. Why is reverse bias useful?

e. Sketch the current produced in the silicon photodiode as a function of wavelength at constant irradiance. Discuss the plot briefly. It may help you to know that the bandgap of silicon is 1.1 eV.

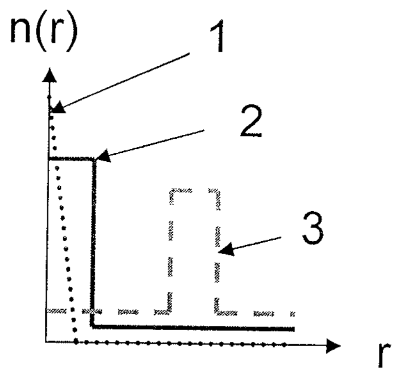
(10 points total) An *f*/2 optical system is operating with incoherent light at a nominal wavelength of 500 nm. The system has 1 wave of astigmatism at the edge of the field of view.

A. (3 points) Assuming paraxial focus, sketch the ray fans for an on-axis object, an object at the edge of the field of view, and an object halfway in between.

B. (3 points) Where are the tangential and sagittal foci for this system relative to paraxial focus?

C. (4 points) Sketch the MTF (without computing it) in both the radial and tangential directions at these same three locations assuming paraxial focus. Qualitatively compare the important features of the MTF (e. g. cutoff frequency) to the diffraction limited MTF. Be sure to label the MTF in physical units related to the problem statement.

(a) The following graphs show the index profile for three types of optical fibers. Sketch the intensity profile I(r) of the lowest order mode for each of the fibers. Which mode has the smallest mode diameter? (20%)



(b) What is the cutoff wavelength of a single mode step index fiber with core diameter of 8 micron, core index of 1.5, and cladding index of 1.496? (20%)

(c) What is modal dispersion? Which type of fiber has the largest modal dispersion? (20%)

(a) step index multimode
(b) graded index multimode
(c) step index single mode
(d) dispersion shifted single-mode
(e) polarization maintaining

(d) You lose 1 dB coupling a 1 mW light source into an optical fiber. You need a signal of 0.1 mW at the other end. How far can you send a signal through fiber with attenuation of 0.5dB/km? You can assume that there is no loss for output coupling. (20%)

(e) A single mode fiber has material dispersion of 20 ps/nm-km and waveguide dispersion of -15 ps/nm-km at the signal wavelength. You send 200 ps pulses through a 100 km length of the fiber using a laser with spectral width of 0.002 nm and wavelength of 1.55 micron. You can assume a transform-limited pulse with time bandwidth product, $\Delta f \tau = 1/2$. What is the maximum data rate you can use? You can assume the period is twice the pulse width. (20%)

The central slice theorem critical to the description of computed tomography can be stated in the following mathematical form:

$$\mathcal{F}_1\{p_\phi(t)\}|_{t\to k} = P_\phi(k) = \mathcal{F}_2\{f(\mathbf{r})\}|_{\mathbf{r}\to\mathbf{k}=\hat{\mathbf{n}}k} = F(\hat{\mathbf{n}}k). \quad (1)$$

(5 points) Describe the meaning of the following symbols in this equation

- $\mathbf{r}$: (example answer — 2D vector describing spatial position in the object)
- $\mathcal{F}_1$:
- $\mathcal{F}_2$:
- $f(\mathbf{r})$:
- ϕ :
- t :
- $p_\phi(t)$:
- k :
- $t \to k$:
- $\mathbf{k}$
- $\hat{\mathbf{n}}$:

(5 points) Using the notation above write down the forward Radon Transform defining the mathematical relationship between $p_\phi(t)$ and $f(\mathbf{r})$. Show a diagram to explain the meaning of your equation.

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$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

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$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

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$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

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$\oint_S (\mathbf{F} \cdot \mathbf{n})\, da = \int_V (\nabla \cdot \mathbf{F})\, d^3x$

$\oint_C \mathbf{F} \cdot d\ell = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da$

$\oint_S \phi\mathbf{n}\, da = \int_V \nabla\phi\, d^3x$

$\oint_S \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_V [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

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System of units: MKSA

A linearly-polarized, monochromatic plane-wave propagates along the x-axis, its E-field amplitude being $\boldsymbol{E}(x,t)=E_o\cos\{\omega[t-n(\omega)x/c]\}\hat{y}$. The host medium is a homogeneous, isotropic, non-magnetic (i.e., $\mu=\mu_o$), transparent dielectric, whose frequency-dependent refractive index is specified as $n(\omega)=\sqrt{\varepsilon(\omega)}$.

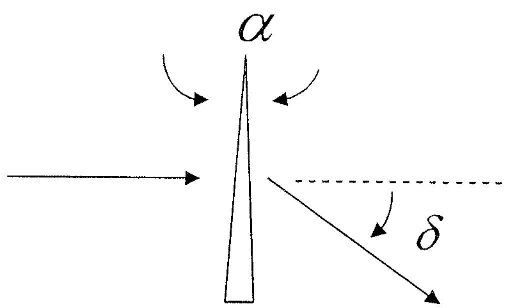
(3 pts) a) Find the magnetic field $\boldsymbol{H}(x,t)$ of the plane-wave in terms of E_o, c, ω, $n(\omega)$, and the impedance of the free space $Z_o=\sqrt{\mu_o/\varepsilon_o}$.

(3 pts) b) Find the Poynting vector $\boldsymbol{S}(x,t)$ of the above plane-wave, then determine the time-averaged rate-of-flow of optical energy (per unit area per unit time) along the x-axis.

(4 pts) c) Assume a second plane-wave, *identical* with the one above *except* for its frequency ω' differing slightly from ω, is co-propagating with the above plane-wave. Write an expression for the combined E-field of the superposed plane-waves. From this expression, identify the carrier and the envelope of the beat waveform. In terms of c, $\omega_c=\frac{1}{2}(\omega+\omega')$, $\Delta\omega=\omega'-\omega$, $n(\omega_c)$ and $\mathrm{d}n(\omega)/\mathrm{d}\omega$, what is the *phase* and *group* velocity of the combined waveform?

THE OPTICAL WEDGE

Consider an optical wedge with wedge angle α and index of refraction n_d as shown:



Optical wedge. The angle α is the angle of the wedge.

1) Determine the angular deviation δ of a beam of light at the d wavelength that passes through the wedge at n_d. 2 points

2) Determine the dispersion for rays of light at the F and C wavelengths at n_F and n_C. 2 points

3) If two wedges, A and B, with wedge angles α and β, are made out of glasses with Abbe numbers ν_A and ν_B, then determine the individual wedge deviation to have a net deviation of δ_{NET} for d light but no dispersion between light at the F and C wavelengths. Make a drawing of the situation. 5 points

4) In the wedge system of part 3 how does the dispersion change if the prism system is tilted relative to the input light beam? Explain. 1 point.

In this problem, you will determine if a stable 2-mirror optical cavity containing a 3-level homogeneous gain medium will have sufficient gain to lase for a given set of conditions. Assume that a resonant mode of the optical cavity (ν_q) is centered on the atomic transition frequency ν_0 between levels 1 and 2, so that we can ignore the frequency dependence of the gain coefficient for this problem.

a. **(2 pts)** For the 3-level system shown in the figure below, set up the rate equations for the population densities $N_1, N_2,$andN_3. Then, derive the expression for the small signal steady-state population density inversion $\Delta N_o = N_2 - N_1$ between levels 1 and 2 in terms of the pumping rate P, the spontaneous emission rate (Γ_{21}), and total number density in the 3 level system, $N_t = N_1 + N_2 + N_3$. Assume the system is below the lasing threshold, such that the intracavity field is negligible and the populations change only due to spontaneous emission and pumping at rate P. Also assume the population decays instantaneously from level 3 to 2 ($N_3 = 0$).

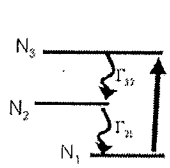
b. **(1 pt)** For the optical cavity shown below, consider an initial field with on axis intensity I_o at position $z = z_1$ that propagates back and forth once in the cavity to complete a round trip. Assume the gain is sufficiently small that we are *below* the threshold for lasing, such that $I_{rt} < I_o << I_{sat}$, where I_{sat} is the saturation intensity of the gain medium. Write an expression for the intensity after one round trip, I_{rt}, in terms of I_o, the mirror power reflectivity coefficients R_1 and R_2, and the small signal gain coefficient at line center, γ_o (there are no losses in the cavity other than from the mirrors). We will not be concerned with the transverse mode profile of the field in this problem, so assume the transverse mode profile is an eigenstate of the cavity.

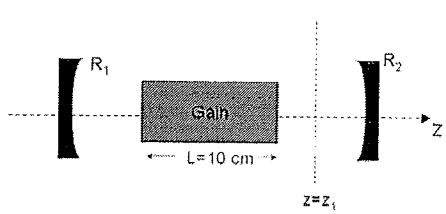
c. **(1 pt)**By increasing γ_o, the cavity will eventually lase. For a continuous-wave laser operating *just above the lasing threshold and in the steady-state* , what numeric value will the ratio I_{rt}/I_o be equal to? You should no longer assume $I_{rt} < I_o << I_{sat}$.

d. **(3 pts)** Recalling that the small signal gain coefficient at the center of the atomic transition can be expressed as $\gamma_o = \sigma_o \times \Delta N_o$, where σ_o is the on resonance absorption cross-section, determine whether or not the laser will lase given the parameters below. You will need to use your results from parts (a) and (b). Be sure to show your work.
$\sigma_o = 10^{-12} cm^2, N_t = 3 \times 10^9/cm^3, P = 2 \times \Gamma_{21}, R_1 = 1, R_2 = .99$

e. **(3 pts)**Assume the laser pumping rate P is sufficiently large that it is now operating above threshold. If P is increased further, briefly describe but do not calculate what will happen (if anything) to the following quantities in the steady state:

i) the small signal gain coefficient γ_o?
ii) the gain coefficient γ?
iii) the laser output power?





Suppose that an atom interacts with a monochromatic laser field $E = \frac{1}{2}\hat{\epsilon}E_0 e^{-i\omega t} + c.c.$, where ω is the optical frequency and $\hat{\epsilon}$ is the polarization vector. This field couples the bare atomic states $|\psi_1\rangle$ and $|\psi_2\rangle$. In the presence of the laser field, the 2-level atom approximation can be used to write the time-dependent atomic state as:

$$|\Psi(t)\rangle = c_1(t)|\psi_1\rangle e^{-iE_1 t/\hbar} + c_2(t)|\psi_2\rangle e^{-iE_2 t/\hbar},$$

with $\omega_0 = (E_2 - E_1)/\hbar$ the atomic resonance frequency for this transition. For the questions below, assume the following: (1) the electric dipole approximation and the rotating wave approximation are valid; (2) spontaneous emission can be neglected; (3) the semi-classical model of light-matter interaction is valid; and (4) the atomic electric dipole moment is parallel with $\hat{\epsilon}$.

(a) Using the variables defined above, write mathematical expressions that define the dipole matrix element $\wp$, the on-resonance Rabi frequency Ω_0, and the generalized Rabi frequency Ω. *[2 points]*

(b) Suppose that $c_1(0) = 1$ and $\omega = \omega_0$. Plot $|c_2(t)|^2$ vs. time for the range of times $0 \leq t \leq 4\pi/\Omega_0$. Label your plot axes, and on each axis, label the range of values associated with this plot (ie, indicate maximum and minimum plotted values for $|c_2(t)|^2$ and t). *[2 points]*

(c) Plot the expectation value of the internal atomic energy vs. time such that it corresponds to your sketch of part (b). Assume the same initial conditions and time range as in part (b), and neglect the energy of the laser field. Label your plot. *[1 point]*

(d) Describe the two main ways in which the Rabi oscillations will change as ω is detuned away from ω_0. You do not need to write any mathematical expressions for this part; a brief description is fine. *[2 points]*

(e) Now assume again that $c_1(0) = 1$ and $\omega = \omega_0$, but suppose that first-order time-dependent perturbation theory is used instead of the 2-level atom approximation to determine $c_2(t)$. **Without doing any calculations**, use your physical intuition to **qualitatively** sketch $|c_2(t)|^2$ as would be predicted by first-order time-dependent perturbation theory. **Draw your sketch as a dotted line directly on the plot you made for part (b), and only sketch $|c_2(t)|^2$ for a range of times $0 \leq t \leq \frac{1}{2}\pi/\Omega_0$.** *[1 point]*

(f) *Briefly* mention the conditions for which first-order time-dependent perturbation theory results are valid in a problem like this one. *[1 point]*

(g) Suppose that the atom is cold enough that it has a deBroglie wavelength of a few microns, while the wavelength of the laser light is less than one micron. Is the electric dipole approximation still valid? Justify your answer. *[1 point]*

Consider interferometric tests to measure temporal coherence characteristics of a fiber laser, which is spatially coherent, exhibits $\bar{\lambda} = 1050\text{nm}$ and has a power spectrum width of approximately $\Delta\nu = 10 \times 10^9 \text{Hz} = 10\text{GHz}$. In the designs below, include travel and/or observation ranges required. State any assumptions that you make.

a.) (4 pts) Design a test of temporal coherence using a Twyman-Green interferometer. The object of the test is to verify the width of the power spectrum.

a) (4 pts) An alternative to the Twyman-Green interferometer is a Young's double pinhole interferometer (YDPI). Is it practical to use a YDPI for this test? Justify your answer.

c.) (2 pts) Without changing your designs, if the laser exhibits multiple-longitudinal-mode characteristics with $\Delta\nu_{\text{mode separation}} \gg \Delta\nu$ and each mode having a width of $\Delta\nu$, qualitatively explain how the experimental results would change for both interferometers.

There is no need for excessive formulae to answer the six parts of this question, you may simply state equations you feel are relevant.

(a - 1pt) Write down the ray transform that relates the incident ray vector $\binom{x}{x'}_i$ to the final ray vector $\binom{x}{x'}_f$ across a first-order optical system.

(b - 2pts) For the case that the ray transfer matrix elements $A = m > 0, B = 0$, provide a ray optical explanation of the relation between the transverse spatial structure of the output beam profile of the corresponding first-order optical system and the incident beam profile.

(c - 1pt) For coherent illumination, describe an optical system other then free-space propagation whose output transverse intensity is proportional to the modulus squared of the Fourier transform of the input beam.

(d - 2pts) Sketch the layout of a Twyman-Green interferometer and state whether or not it is based on division of amplitude or wavefront.

(e - 2pts) Write down the equation of motion for the Lorentz electron oscillator describing an atom in interaction with a light field and clearly identify the terms that incorporate the effects of transverse and longitudinal electric fields.

(f - 2pts) Sketch the characteristic arrangement of the incident optical field wavevector $\vec{k}_i$, acoustic wavevector $\vec{K}$, and the scattered field components for both the Bragg diffraction and Raman-Nath diffraction geometries.

Assume the absorption coefficient of a two-band semiconductor to be given by

$$\alpha(\omega) = \text{const.}\,|\,d_{cv}(0)\,|^2 \left(\frac{2m_r}{\hbar^2}\right)^{3/2} \theta(\hbar\omega - E_g)\sqrt{\hbar\omega - E_g} \qquad (1)$$

where "const." denotes a constant and Coulomb effects have been neglected.

(a) Sketch the frequency dependence of the absorption coefficient, properly labeling the axes and all special points.

(2 points)

(b) Using the appropriate quantity from Eq. (1), give a mathematical statement expressing the fact that the optical transition is first-class dipole allowed.

(2 points)

(c) Assuming you can modify the electron mass without changing any other property of the semiconductor, how do you have to modify the electron mass in order to increase the absorption coefficient at a fixed frequency outside the transparency region?

(2 points)

(d) Assuming you can modify the hole mass without changing any other property of the semiconductor, how do you have to modify the hole mass in order to increase the absorption coefficient at a fixed frequency outside the transparency region?

(1 point)

(e) Define the frequency interval that corresponds to the transparency region. Which of the following parameters can be used (and if so, how do you have do modify them?) to increase the transparency region: (i) the electron mass, (ii) the hole mass, (iii) the optical dipole matrix element, and (iv) the band gap energy. Sketch the band structure underlying Eq. (1), properly labeling all axes, special points and the bands.

(3 points)

1. Answer the following questions about the emission of radiation from accelerated charge:
   a. Write an equation for the current density $\underset{\sim}{J}(\underset{\sim}{r}',t')$ associated with a single electron at location $\underset{\sim}{r}'$ at time t' moving with non-relativistic velocity $\underset{\sim}{v}(t')$. Demonstrate that the units work out.

   b. In the expression for a vector potential arising from a current density,

$$\underset{\sim}{A}(\underset{\sim}{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\left[\underset{\sim}{J}(\underset{\sim}{r}',t')\right]_{ret}}{\left|\underset{\sim}{r}-\underset{\sim}{r}'\right|},$$

   explain what is meant by the $[\ ]_{ret}$ operator. Give a physical interpretation for the variables $\underset{\sim}{r}$, $\underset{\sim}{r}'$, t, and t'.

   c. The electric field $\underset{\sim}{E}(\underset{\sim}{r},t)$ can be written in terms of this vector potential $\underset{\sim}{A}(\underset{\sim}{r},t)$ and a scalar potential $\Phi(\underset{\sim}{r},t)$ via

$$\underset{\sim}{E}(\underset{\sim}{r},t) = -\frac{\partial \underset{\sim}{A}(\underset{\sim}{r},t)}{\partial t} - \nabla\Phi(\underset{\sim}{r},t).$$

   Explain which term is responsible for the emission of light and how the dependence on the acceleration of charge arises.

   d. Draw the radiation pattern based on the orientation of the acceleration vector $\dot{\underset{\sim}{v}}(t')$ (a 2D sketch on a plane containing $\dot{\underset{\sim}{v}}(t')$ is adequate). Label the figure, including what is being plotted.

Assume the absorption coefficient of a two-band semiconductor to be given by

$$\alpha(\omega) = \text{const.}\, |d_{cv}(0)|^2 \left(\frac{2m_r}{\hbar^2}\right)^{3/2} \theta(\hbar\omega - E_g)\sqrt{\hbar\omega - E_g} \qquad (1)$$

where "const." denotes a constant and Coulomb effects have been neglected.

(a) Sketch the frequency dependence of the absorption coefficient, properly labeling the axes and all special points.

(2 points)

(b) Using the appropriate quantity from Eq. (1), give a mathematical statement expressing the fact that the optical transition is first-class dipole allowed.

(2 points)

(c) Assuming you can modify the electron mass without changing any other property of the semiconductor, how do you have to modify the electron mass in order to increase the absorption coefficient at a fixed frequency outside the transparency region?

(2 points)

(d) Assuming you can modify the hole mass without changing any other property of the semiconductor, how do you have to modify the hole mass in order to increase the absorption coefficient at a fixed frequency outside the transparency region?

(1 point)

(e) Define the frequency interval that corresponds to the transparency region. Which of the following parameters can be used (and if so, how do you have do modify them?) to increase the transparency region: (i) the electron mass, (ii) the hole mass, (iii) the optical dipole matrix element, and (iv) the band gap energy. Sketch the band structure underlying Eq. (1), properly labeling all axes, special points and the bands.

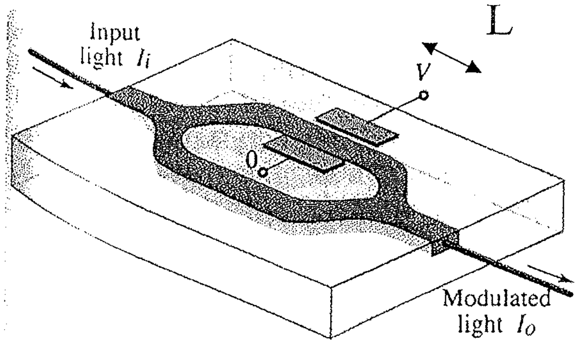
(3 points)

A radiative cooler for a spaceborne instrument has a square radiator 0.5 meters on a side. The broadband emissivity of the radiator is e = 0.9. The back surface of the radiator is insulated. The spaceborne instrument must dissipate 9 W and needs to be at a temperature of 200 K or colder. What is the equilibrium temperature of the radiator when it dissipates 9 W into space (4 pts.)?

If the spacecraft turns so that solar radiation is now falling on the radiator, its temperature will rise. The spacecraft is in low Earth orbit and the Sun subtends a full angle of 0.5°. Model the Sun as a blackbody with temperature of 6,000 K.
Plot the equilibrium temperature of the radiator as a function of angle of incidence of solar irradiance and determine the angle at which the equilibrium temperature is 200 K (4 pts.).

Explain how the performance of this radiative cooler can be improved in the presence of sunlight. Provide one spatial and one spectral option (2 pts.).

Planar waveguide modulator



(a) The diagram shows a planar waveguide modulator that operates by the electro-optics effect. Explain what is the electro-optics effect and how does the modulator work? (25%)

(b) The index of the substrate is given by $n(E) = n_0 - \frac{1}{2} r n^3 E$, where r is the Pockels coefficient and E is an applied electric field. What is the expression for the phase shift difference $\Delta\varphi$ between the two arms if the cell length is L? (25%)

(c) Assuming $E = V / d$, where d is some effective channel width and V is the applied voltage, determine the minimum voltage necessary to go from maximum to minimum output intensity. (25%)

(d) What is the functional relationship between the output intensity I_o to the input intensity I_i and $\Delta\varphi$? Where does the light go when the output light is at the minimum? (25%)

Koehler illumination is the most common type of illumination system used in optical microscopes.

a. (4 points) Show a diagram of a Koehler illumination system for brightfield microscopy. Be sure to show the source (lamp), lenses, apertures, and object plane as well as critical distances that separate them.

b. (2 points) Explain the aperture size adjustments in the Koehler illuminator (i.e. what apertures and for what purpose are aperture sizes adjusted).

c. (2 points) Explain how misadjustment of the Koehler illuminator condenser aperture size can affect the spatial resolution of the imaging system.

d. (2 points) Describe how the Koehler illumination system is modified for dark-field microscopy.