Optical Sciences PhD Qualifying Exam: Instructions and Equation Sheet

Answer all four questions on the following pages. Start each answer on a new sheet of paper.

In the upper right hand corner of each sheet, write your code number (NO NAMES PLEASE) and the problem number (course number) that you are answering. Attach together all sheets for a given problem in order.

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$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}$

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$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

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$\sin^2(\frac{A}{2}) = \frac{1}{2}(1 - \cos A)$

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$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$\nabla\phi\psi = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\phi$

$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

$\nabla \cdot (\nabla \times \mathbf{F}) = 0$

$\nabla \times (\phi\mathbf{F}) = \phi(\nabla \times \mathbf{F}) + \nabla\phi \times \mathbf{F}$

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$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

$\nabla \times \nabla\phi = 0$

$\oint_{\mathrm{S}} (\mathbf{F} \cdot \mathbf{n})\, da = \int_{\mathrm{V}} (\nabla \cdot \mathbf{F})\, d^3x$

$\oint_{\mathrm{C}} \mathbf{F} \cdot d\ell = \int_{\mathrm{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da$

$\oint_{\mathrm{S}} \phi\mathbf{n}\, da = \int_{\mathrm{V}} \nabla\phi\, d^3x$

$\oint_{\mathrm{S}} \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_{\mathrm{V}} [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

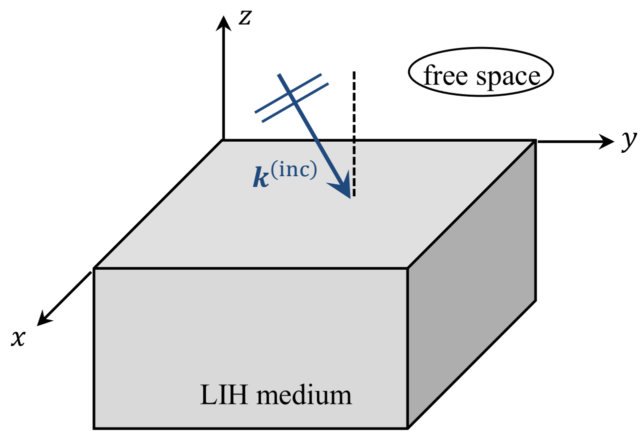
$\oint_{\mathrm{S}} (\mathbf{n} \times \mathbf{F})\, da = \int_{\mathrm{V}} (\nabla \times \mathbf{F})\, d^3x$

PhD Qualifying Exam, Summer 2023

Opti 501, Day 1

<u>System of units: SI (or MKSA)</u>

Suppose a homogeneous plane-wave (i.e., one whose k-vector is real) arrives from the free space onto the flat and polished surface of a linear, isotropic, and homogeneous (LIH) medium, as shown. Let the interface between the LIH medium and the medium of incidence (i.e., free space in the present case) be the xy-plane at $z = 0$. Using brief but precise statements, define the following properties of the optical material, characteristics of the plane-wave, and specific features of the optical system.



a) When is an optical medium considered to be linear, isotropic, and homogeneous (LIH)?

b) What is the plane of incidence? Does this definition hold for a normally-incident plane-wave?

c) When is the incident plane-wave said to be p-polarized? When is it said to be s-polarized?

d) Denoting the components of the incident E-field by $E_p = |E_p|e^{\mathrm{i}\varphi_p}$ and $E_s = |E_s|e^{\mathrm{i}\varphi_s}$, describe conditions under which the incident plane-wave can be said to be linearly polarized, or circularly polarized, or elliptically polarized.

PhD Qualifying Exam, Fall 2023
OPTI 502, Day 1

Design an infinity-corrected microscope objective with a magnification of 4X, and an object numerical aperture of 0.1. Do not use the small angle approximation for numerical aperture.

The working wavelengths correspond to the spectral F, d, and C lines. The focal length of the tube lens is 200 mm, and the object, half height, is 2 mm.

The stop aperture must be placed at the rear focal point of the objective; the tube lens is located at the stop aperture. Assume thin lenses for the objective and the tube lens to answer the following questions.

1. Make a neat drawing of the objective lens and the tube lens showing the marginal and chief rays. (1 Point)
2. What is the focal length of the objective? (1 point)
3. What is the diameter of the aperture stop? (1 point)
4. What is the advantage of placing the stop aperture at the rear focal point of the objective? (1 point)
5. If N-BaK7 (n=1.52, V=64.2) is the only material you can use, what is the radius of curvature of the surface for a plano-convex singlet used as the objective lens? (1 point)
6. What is the axial chromatic aberration of the objective lens? (1 point)
7. Which material, N-SF2 (n=1.65, V=33.8) or N-SK16 (n=1.62, V=60.3), would you choose to design an achromatic doublet with N-BK7 for correcting axial chromatic aberration? Why? (2 points)
8. Provide the focal length of each element of the thin achromatic doublet using N-BK7 glass and your glass selection in problem 7. (2 points)

Ph.D. Qualifying Exam, Fall 2023

Day 1

OPTI 505R

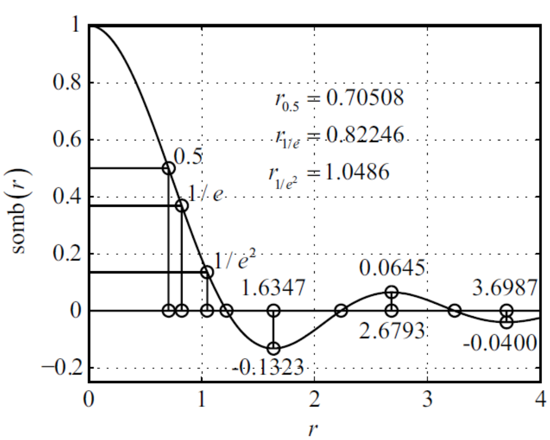
1.) (2pts) A cell phone camera has a focal length of 5 mm and a circular pupil diameter of 2.5 mm. You may assume that the distance from the pupil to the image plane is equal to the focal length. If there are no aberrations, what is the zero-to-zero Airy disk diameter for a wavelength of 550nm?

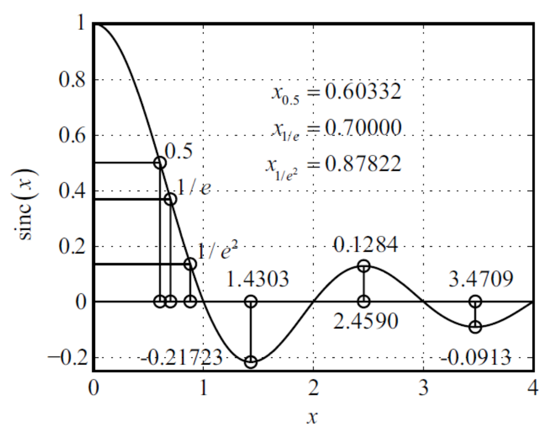
2.) (2pts) What is the modulation transfer function (MTF) incoherent cutoff frequency of (1) for a wavelength of 550nm? Express your numerical answer in lines/mm.

3.) (2pts) (Yes/No) Does changing the aperture shape from circular to square change the MTF incoherent cutoff frequency? Assume that the analysis is done in the x dimension, sides of the square are aligned with the x and y axes, and the length of the side of the square is the same as the dimeter of the circular pupil in (1). Explain your reasoning.

4.) (2pts) The lens in (1) is used to image a light-emitting diode (LED) that is 1 mm in diameter at some distance away from the lens. What is the smallest distance z for which the lens is illuminated with spatially coherent light across the diameter of the pupil? You may assume that the LED is an extended quasi-monochromatic source with wavelength 550nm.

5.) (2pts) What is the diameter of the LED's geometrical image using the distance that you have chosen in (4), and how does it compare to the Airy disk diameter in (1)?





Fall 2023 Qualifying Exam
Day 1
OPTI 511R / 570/ 544
(10 points total)

Consider a particle of mass m confined to a 1D harmonic potential of frequency ω that is centered on the origin at $x = 0$. Denote the energy eigenstates of the system in Dirac notation as $|n\rangle$ with associated energy eigenvalues E_n, and $n = 0, 1, 2, \ldots$. See the bottom of this page for information you may find useful.

1. What are the energy eigenvalues of the ground state $|0\rangle$ and second excited state $|2\rangle$? (1 pt)

2. Sketch the wavefunctions of $|0\rangle$ and $|2\rangle$ in position space versus $(x/\sigma) = [-3, 3]$, where $\sigma = \sqrt{\frac{\hbar}{m\omega}}$ is the ground state width. Sketch both wavefunctions on the same plot and highlight their variation along the horizontal axis (x/σ), the absolute scale along the vertical axis being less important. Choose overall phase factors such that both wavefunctions are real and positive at the origin $x = 0$. (2 pts)

3. Suppose that at time $t = 0$, the particle is in an initial state

$$|\Psi(0)\rangle = c_0|0\rangle + c_2|2\rangle$$

where c_0 and c_2 are real and positive and $c_2 \ll c_0 \approx 1$. Argue graphically that, in position space, the wavefunction of $|\Psi(0)\rangle$ resembles a slightly compressed version of the ground state. (2 pts)

4. Provide an argument that the initial expectation value of the particle position is $\langle\hat{x}\rangle(t = 0) = \langle\Psi(0)|\hat{x}|\Psi(0)\rangle = 0$ (this result holding for all times, $\langle\hat{x}\rangle(t > 0) = 0$). (1 pt)

5. Without requiring a derivation, give an expression for the state vector $|\Psi(t)\rangle$ at a later time $t > 0$ in terms of c_0, c_2, $|0\rangle$, $|2\rangle$ and ω. (1 pt)

6. Based on the above information derive the following expression for the time-dependent expectation value of the particle position squared

$$\langle\hat{x}^2\rangle(t) = \langle\Psi(t)|\hat{x}^2|\Psi(t)\rangle = \frac{\sigma^2}{2}\left[1 - 2\sqrt{2}c_2\cos(2\omega t)\right],$$

correct to first-order in c_2: That is, since $c_2 << 1$ you may drop terms involving c_2^2 and set $c_0 = 1$. You will also need the matrix elements $\langle 0|\hat{x}^2|0\rangle = \frac{\sigma^2}{2}$ and $\langle 0|\hat{x}^2|2\rangle = \langle 2|\hat{x}^2|0\rangle = -\frac{\sigma^2}{\sqrt{2}}$. (2 pts)

7. Bringing these results together, show that for $c_2 << 1$ the root-mean-square particle position $\Delta x(t) = \sqrt{\langle\hat{x}^2\rangle(t) - \langle\hat{x}\rangle^2(t)}$ oscillates with frequency 2ω around the value for the ground state. (Even if you have trouble with part 6 you can still use the result given there to address this part.) (1 pt)

$$\psi_0(x) \propto \exp(-x^2/(2\sigma^2))$$
$$\psi_1(x) \propto (x/\sigma)\exp(-x^2/(2\sigma^2))$$
$$\psi_2(x) \propto (2 - 4x^2/\sigma^2)\exp(-x^2/(2\sigma^2))$$

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PhD Qualifying Exam, Summer 2023

Opti 501, Day 2

System of units: SI (or MKSA)

a) For an electromagnetic plane-wave residing inside a linear, isotropic, and homogeneous medium, write a (generally complex) pair of expressions for the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ and the magnetic field $\boldsymbol{H}(\boldsymbol{r}, t)$ in terms of $\boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{k}$, and ω.

b) Assuming that $\rho_{\text{free}}(\boldsymbol{r}, t) = 0$ and $\boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) = 0$, write all four equations of Maxwell for the E-field and H-field of the plane-wave, keeping in mind that the remaining sources within the medium should be expressed as $\boldsymbol{P}(\boldsymbol{r}, t) = \varepsilon_0 \chi_{\text{e}}(\omega) \boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{M}(\boldsymbol{r}, t) = \mu_0 \chi_{\text{m}}(\omega) \boldsymbol{H}(\boldsymbol{r}, t)$. Simplify these equations by eliminating the $\boldsymbol{\nabla} \cdot$, $\boldsymbol{\nabla} \times$, $\partial/\partial t$ operators and using the material medium's (relative) permittivity $\varepsilon(\omega) = 1 + \chi_{\text{e}}(\omega)$ and permeability $\mu(\omega) = 1 + \chi_{\text{m}}(\omega)$.

c) Use Maxwell's 1st equation obtained in part (b) to express E_{0z} in terms of $E_{0x}, E_{0y}, k_x, k_y, k_z$.

d) Derive the dispersion relation $\boldsymbol{k} \cdot \boldsymbol{k} = (\omega/c)^2 \, \mu(\omega) \varepsilon(\omega)$ from the Maxwell equations obtained in part (b). How is the material medium's (complex) refractive index $n(\omega)$ related to its (relative) permittivity $\varepsilon(\omega)$ and (relative) permeability $\mu(\omega)$?

Hint: The vector identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$ should be helpful.

PhD Qualifying Exam, Fall 2023
OPTI 502, Day 2

1) Design a telescope with the following specifications. (10 pts total)

   - Magnifying Power (MP) = 10.
   - Two positive thin lenses (objective and eye lens).
   - Distance between the two lenses = 220 mm.
   - Aperture stop located at the first lens, D_{STOP} = 40 mm .
   - Object at infinity.
   - +/- 3 degree unvignetted field of view in object space.

   a) What are the focal lengths of each lens? (1 pt)

   b) What is the eye relief (distance from eye lens to exit pupil), and size of exit pupil? (2 pt)

   c) What is the diameter of the eye lens to have no vignetting over the field of view? (1 pt)

   d) Add a field lens to the telescope with a focal length of 50 mm. Where is the field lens placed? (1 pt)

   e) With the addition of the field lens, what is the new unvignetted field of view in object space (in degrees) assuming the diameter of the eye lens from part c) does not change? (2 pt)

   f) What is the minimum diameter of the field lens to support the unvignetted field of view that you found in part e)? (1 pt)

   g) What is the minimum angular separation of two stars that are resolvable with this system when coupled to the human eye (which has a resolution of 1 arc min)? (1 pt)

   h) Draw a diagram of the telescope with the objective, eye lens, and field lens, and draw the marginal and chief rays. (1 pt)

Ph.D. Qualifying Exam, Fall 2023

Day 2

OPTI 505R

Two laser beams are generated by passing a collimated laser through a linear grating at normal incidence, as shown below in the schematic figure. The two laser beams we use are the +/-1st orders diffracted from the grating. Other diffracted orders, if any, are blocked. The mirrors are placed with their reflective surfaces at 67.5° with respect to the x axis, as shown. The paths are balanced, so that the distances along the +/- 1st orders between the grating to the mirrors are exactly the same. The horizontal (x) distance between the mirrors is 100mm. x = 0 is in the center, between the mirrors. The beam diameter is a few millimeters, which is large enough to observe fringes, if any. You may assume that any beams reflected from the grating do not contribute to interference. State any assumptions that you make.

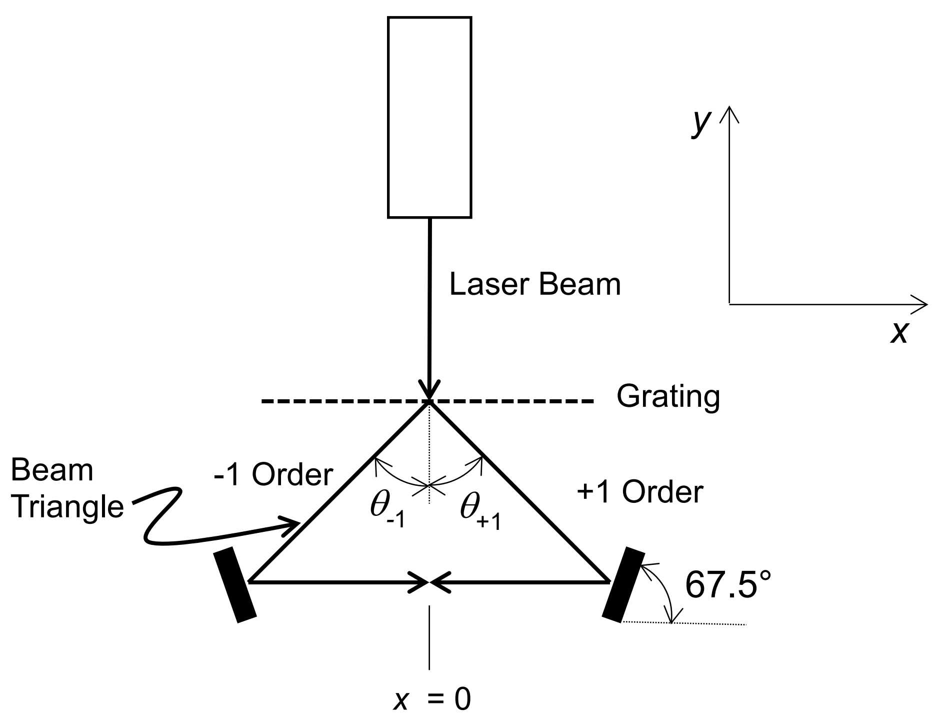
1) (2 pts) If the wavelength of the laser is 500nm and $\theta_{+/-1}$ = +/- 45°, what is the period of the grating?

2) (2 pts) If the laser beam is collimated, monochromatic and is linearly polarized out of the plane of the drawing (z direction), determine where, if any, interference fringes are observed in the beam paths defined by the triangle.

3) (2 pts) If the laser beam is collimated, has a bandwidth of 10 GHz, and is linearly polarized out of the plane of the drawing (z direction), determine where, if any, interference fringes are observed in the beam paths defined by the triangle. Include the range over which fringes can be observed with reasonable visibility. You can assume that the diffracted angle from the grating does not change over this bandwidth. State any other assumptions that you make.

4) (2 pts) For the combined beam in (2), what is the fringe shape and orientation, if any, in the region between the mirrors? What is the fringe spacing?

5) (2 pts) If in (2) the collimated, monochromatic laser beam wavelength changes to 550 nm, what is the fringe shape and orientation, if any, in the region between the mirrors? Is the fringe spacing, if any, larger or smaller than (4)? You do not need to calculate fringe spacing.



Fall 2023 Qualifying Exam
Day 2
OPTI 511R / 570 / 544
(10 points total)

In parts (a)-(c) of this problem we consider a hydrogen atom initially in the ground state $\psi_{1,0,0}$ (associated with quantum numbers $n = 1$, $l = 0$, and $m_z = 0$). Throughout this problem, we neglect electron and proton spins, and we will also neglect spontaneous emission except where explicitly indicated in parts (d)-(e). We also assume that the electric dipole approximation and the rotating wave approximation (or equivalently the resonance approximation) are valid. At time $t = 0$ a monochromatic and linearly polarized laser field $\vec{E} = \hat{z}\frac{1}{2}E_0 e^{-i\omega t} + c.c.$ is turned on. The resonant frequency of the $n = 1$ to $n = 3$ transition is ω_0, and the (small) detuning of the laser from this resonance is given by $\Delta = \omega - \omega_0$. The atom-light coupling of the $n = 1$ to $n = 3$ transition is associated with a resonant (or bare) Rabi frequency Ω_0.

(a) [2 points] Into which state ψ_{nlm} (or $|n, l, m\rangle$) would the atom most likely be excited by the laser? Write down a general expression that gives the probability for finding the atom in this excited state as a function of time (in the absence of spontaneous emission). Clearly define any symbols used in your expression that have not yet been introduced into this problem.

(b) [4 points] Make a plot showing the transition probability into the excited state of part (a) as a function of time, assuming the laser is on resonance ($\Delta = 0$) and has an electric field strength that gives rise to a resonant Rabi frequency of $\Omega_0 = 10^8$ s^{-1}. Be sure to label both axes, and plot from time $t = 0$ to a time $t = t_1$, which is the time needed for a "2π" pulse. On the same graph, make an additional plot from $t = 0$ to $t = t_1$ showing the transition probability under the same conditions except that the laser now has a detuning $\Delta = \sqrt{3}\,\Omega_0$. For this second plot, indicate the maximum transition probably and the transition probability at time t_1. Clearly label each curve with its associated value of detuning.

(c) [1 point] For the case $\Delta = 0$, what is the probability of finding the atom in the excited state after the laser field is turned off, if the laser field is turned off at a time corresponding to a "$\pi/2$" pulse?

(d) [1 point] Assume an ensemble of hydrogen atoms are all initially the state $\psi_{3,1,0}$, and there is no laser field present to induce transitions. What wavelength(s) of light would be emitted from the ensemble as the atoms spontaneously decay? Give numerical value(s). Note: 1 eV $= 1.6 \times 10^{-19}$ J.

(e) [2 points] For the case of part (d), with spontaneous emission occurring, make a plot that shows the fraction of atoms in the state $\psi_{3,1,0}$ as a function of time. Also give the formula that corresponds to this time-dependent fraction of atoms. Your formula should depend on the Einstein A coefficients of the atomic transitions involved in this problem, so let $A_{nlm,n'l'm'}$ denote the A coefficient for an upper state $\psi_{n,l,m}$ and a lower state $\psi_{n',l',m'}$. Use this notation in your answer, and do not attempt to calculate any A coefficients. Define any other new symbols that you use.