TRACK I

WRITTEN PRELIM EXAM – FIRST DAY

Spring 2005

February 14, 2005
8:30 a.m. to 12:00 p.m.

Answer all six questions. Start each answer on a new page.

In the upper right hand corner of each sheet you hand in, put your code number (NO NAMES PLEASE) and the problem number. Staple together all sheets for a given problem.

Insert your answers into the manila envelope supplied and note which problems you have answered on the outside of the envelope.

The following are some helpful items:

$h = 6.625 \times 10^{-34}\ \mathrm{J \cdot s} = 4.134 \times 10^{-15}\ \mathrm{eV \cdot s}$

$e = 1.6 \times 10^{-19}\ \mathrm{C}$

$c = 3.0 \times 10^{8}\ \mathrm{m/s}$

$k_B = 1.38 \times 10^{-23}\ \mathrm{J/K}$

$\tau = 5.67 \times 10^{-8}\ \mathrm{W/K^4 \cdot m^2}$

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}$

$\mu_0 = 1.26 \times 10^{-6}\ \mathrm{H/m}$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$

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$\sin 2A = 2 \sin A \cos A$

$\cos 2A = 2 \cos^2 A - 1$

$\cos 2A = 1 - 2 \sin^2 A$

$\sin^2(\frac{A}{2}) = \frac{1}{2}(1 - \cos A)$

$\cos^2(\frac{A}{2}) = \frac{1}{2}(1 + \cos A)$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$\nabla\phi\psi = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\phi$

$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

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$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

$\nabla \times \nabla\phi = 0$

$\oint_S (\mathbf{F} \cdot \mathbf{n})\, da = \int_V (\nabla \cdot \mathbf{F})\, d^3x$

$\oint_C \mathbf{F} \cdot d\ell = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da$

$\oint_S \phi\mathbf{n}\, da = \int_V \nabla\phi\, d^3x$

$\oint_S \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_V [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

$\oint_S (\mathbf{n} \times \mathbf{F})\, da = \int_V (\nabla \times \mathbf{F})\, d^3x$

SP'25

The formulae for the reflection coefficients for s-polarized and p-polarized incident plane-wave fields at a planar dielectric interface of two media are

$$r_s(\theta) = \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}}, \quad r_p(\theta) = \frac{-n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}},$$

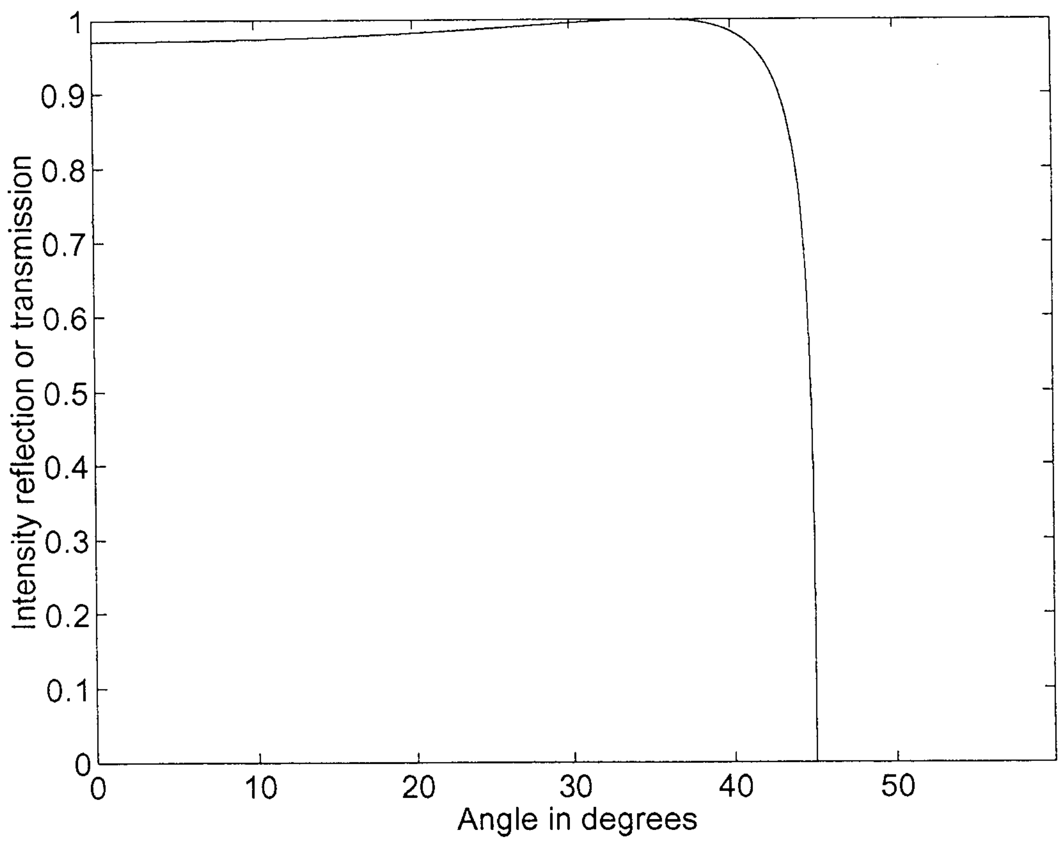
where $n = n_2/n_1$ is the relative refractive index, n_1 being the refractive index of the medium the field is incident from and n_2 the refractive index of the second medium, and θ measures the angle of incidence with respect to the normal to the plane of the interface in the incident medium.

(a) Explain what is meant by the term plane of incidence, and how s-polarized and p-polarized incident fields are defined with respect to the plane of incidence. (2 points)

(b) The wave vector of the plane-wave field of frequency ω incident on the interface is denoted $\vec{k}$, and the reflected and transmitted wave vectors as $\vec{k}'$ and $\vec{k}''$, respectively. By equating the components of each wave vector resolved along the interface, prove that the reflected angle equals the incident angle $\theta' = \theta$, and Snell's law of refraction $n_1 \sin\theta = n_2 \sin\phi$, with ϕ the angle of the transmitted wave vector with respect to the normal to the plane of the interface in the second medium. (3 points)

(c) Using either the expression for $r_s(\theta)$ or $r_p(\theta)$ derive an expression for the critical angle for total internal reflection θ_{cr}, and explain any conditions on the relative refractive-index n to observe total internal reflection. (2 points)

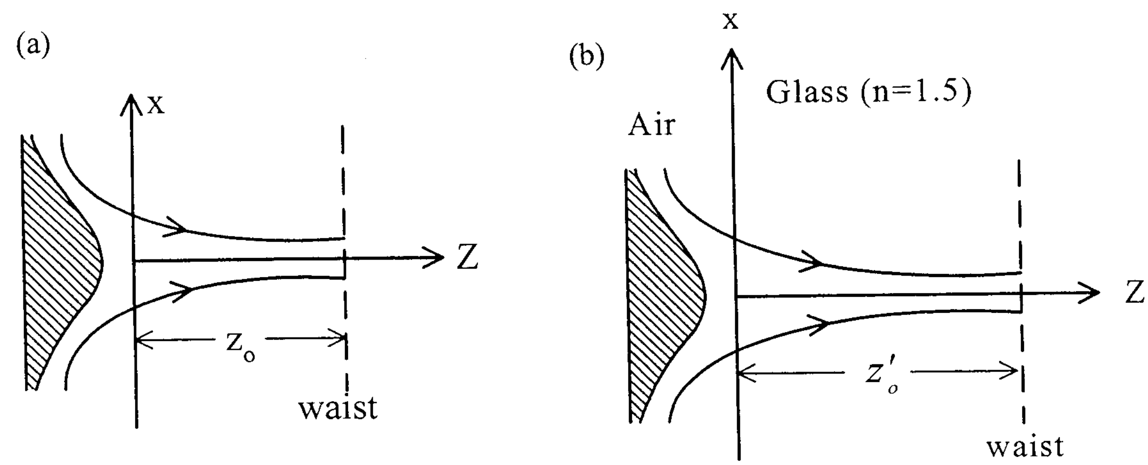
(d) The figure on the next page shows (i) either the intensity reflection R or intensity transmission T, for (ii) either internal of external reflection at a dielectric interface as a function of incident angle θ, and for (iii) either an s-polarized or p-polarized incident field. By inspecting the curve explain to which combination of the above options (i),(ii) & (iii) the plotted curve corresponds. (3 points)


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Intensity reflection or transmission
0
10
20
30
40
50
Angle in degrees

Question 2
Day 1/Track 1
Sp 05

Consider a Gaussian beam with the following parameter values at $z=0$: $\lambda = 0.5\mu m$, $r_o = 1$ mm, $R_c = -10$ mm. [Notation: λ = vacuum wavelength, r_o = 1/e (amplitude) radius, R_c = radius of curvature.]

a) With reference to Figure (a) below, find the distance z_o to the waist, and the 1/e radius of the beam at the waist, $r_o(z_o)$, in units of millimeters.
b) Assume a flat piece of glass (refractive index $n = 1.5$) is placed in front of the beam, as in Figure (b). The flat interface between air and glass is at $z = 0$. Find the new distance to the waist, z'_o, and the new beam radius at the waist, $r'_o(z'_o)$.



Question 3
Day 1/Track 1

Consider a linear shift-invariant (LSI) system with input $f(x)$, output $g(x)$.

a) Show that if $\frac{d}{dx} f(x)$ is input, $\frac{d}{dx} g(x)$ will be output.

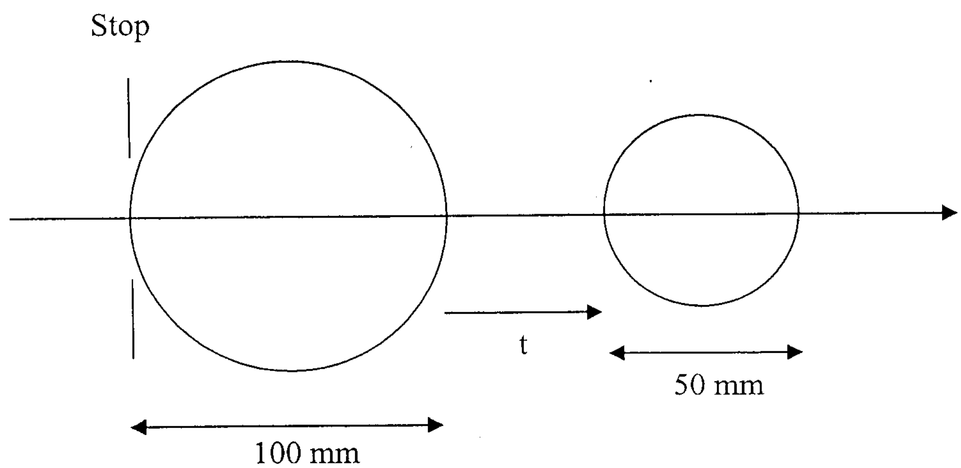
b) Let $f(x) = step(x)$ be used as input, and assume that the corresponding output is $g(x)=\begin{Bmatrix} 0 & x<0 \\ 1-e^{-\alpha x} & x\geq 0 \end{Bmatrix}$. Determine the impulse response $h(x)$ of the system.

c) Again let $f(x) = step(x)$ be used as input. This time assume the corresponding output is $g(x)=\begin{Bmatrix} 0 & x<0 \\ e^{-\alpha x} & x\geq 0 \end{Bmatrix}$. Determine the impulse response $h(x)$ of the system.

Note: α is a positive real number.

Question 4
Day 1/Track 1
Sp '05

Design an afocal telescope using two spherical glass balls. The diameter of the first ball is 100 mm, and the diameter of the second ball is 50 mm. Both balls have an index of refraction of 1.5. The system stop for this telescope is placed at the front surface of the first ball.



Provide the required *separation* of the balls t and determine the *magnifying power* and *eye relief* of the resulting telescope.

Question 5
Day 1/Track 1
Sp '25

(2 points) Write down the functional form of the density of states $g^{(3d)}(\varepsilon)$ in 3D.

(2 points) Write down the functional form of the density of states $g^{(2d)}(\varepsilon)$ in 2D.

(2 points) Calculate the total electron density in the conduction band for a semiconductor at an equilibrium temperature T. Assume the Maxwell Boltzman distribution function for the electrons. Assume the zero of energy is at the top of the valence band.

(2 points) Where is the location of the chemical potential in an intrinsic semiconductor at room temperature.

(2 points) Where is the location of the chemical potential for a highly doped n-type semiconductor.

Question 6
Day 1/Track 1
Sp'05

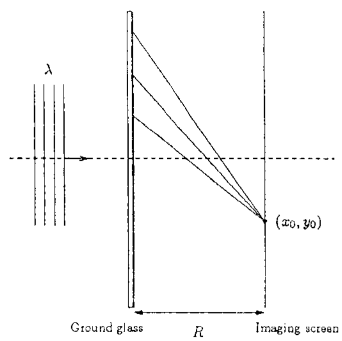


Figure 1

Let a collimated, coherent light beam of wavelength λ be incident on a piece of ground glass as shown in Figure 1. Some distance away from this ground glass is an imaging screen. We are interested in understanding the statistical distribution of the beam intensity and phase measured at location (x_0, y_0) — speckle statistics.

We begin by splitting up the ground glass into N random "scattering spots," where N is a very large number. Each scattering spot emits a Huygens' wavelet that contributes to the complex wavefield at (x_0, y_0). Using Huygens' wavelets and the imaging geometry we can derive a very simple expression for the the complex wavefield u at location (x_0, y_0) as

$$u = K \sum_{n=1}^{N} \exp\left(i\theta_n\right), \tag{1}$$

where K is a constant and θ_n is a random phase caused by small distance variations in the ground glass. The probability density function for θ_n is assumed to be uniform between 0 and 2π and the θ_n's are independent from one another. This expression is somewhat simplified from the actual result but it still illustrates the point.

1. (30%) Using Eqn. 1, determine the probability density functions (univariate, not joint) for the real and imaginary components of the complex wavefield u_{re} and u_{im}, where $u_{\text{re}} = \text{Real}(u)$ and $u_{\text{im}} = \text{Imag}(u)$.
2. (30%) Determine the means and the variances of both u_{re} and u_{im}.
3. (30%) Prove that u_{re} and u_{im} are uncorrelated from one another. Are they also independent?
4. (10%) Describe how you would go about determining the probability density functions for the intensity and phase measured at location (x_0, y_0). (You do not need to determine the probability densities. You only need to describe the necessary steps to compute these two densities.)

Question 7
Day 1/Track 1
Sp '05

The first zone of a Fresnel zone plate has a radius of 0.3 mm. The even zones of the pate are masked with an opaque coating, which does not transmit any light.

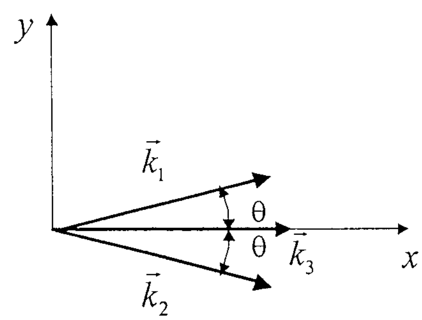
a.) (2 pts.) What is the primary focal length of the zone plate for a wavelength of 900 nm?

b.) (4pts.) The zone plate is illuminated with a converging spherical wave of wavelength 900 nm that focuses 100 mm to the right of the plate. (That is, the object is virtual.) Where do the first two orders (+1 order and -1 order) from the zone plate come to focus?

c.) (4 pts.) All open zones in the plate transmit 100% of the incident light. The zone plate is 6.03 mm in diameter. When illuminated with an on-axis $\lambda = 900$ nm plane wave, how much brighter is the on-axis irradiance at the primary focus than without the zone plate?

Question 8
Day 1/Track 1
Sp'05

Three plane waves propagate as shown below. $\lambda_1 = \lambda_2 = \lambda_3 = 650$ nm. $\theta = 1$ degree. Amplitudes are $a_1 = a_2 = 0.1$ V/m and $a_3 = 1$ V/m. The waves are traveling in water with refractive index $n = 1.33$.



a.) (2 pts.) Assume that the composite wave is purely amplitude modulated at $x = 0$. Write an expression for the total field amplitude as a function of y in the $x = 0$ plane.

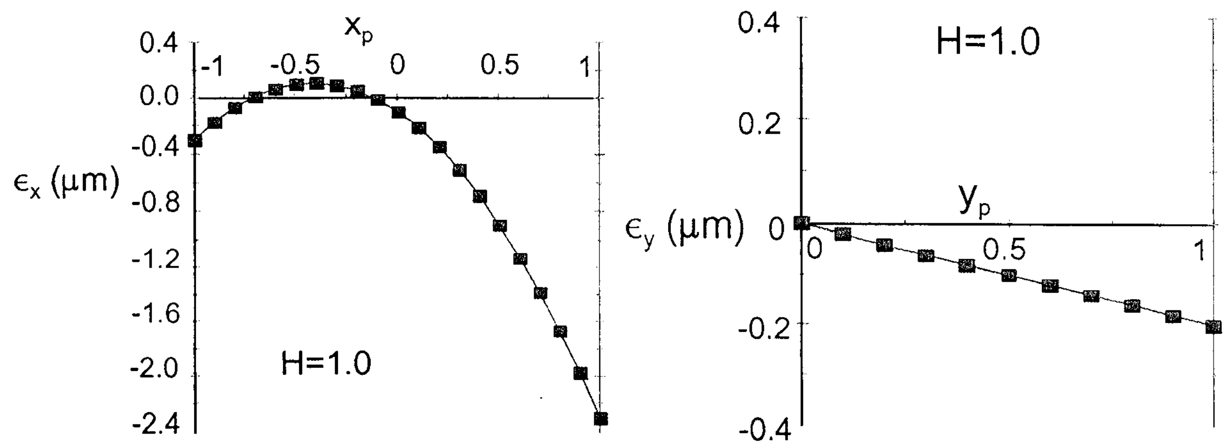
b.) (2 pts.) Write a mathematical expression that describes the plane-wave spectrum for the three waves at $x = 0$.

c.) (2 pts.) What is the distance from $x = 0$ where the composite wave is first purely phase modulated?

d.) (2 pts.) Draw a sketch of the irradiance profile in the y dimension at the plane specified by part (c).

e.) (2 pts.) What is the influence of $\vec{k}_3$ on the irradiance? That is, how is the irradiance different with just $\vec{k}_1$ and $\vec{k}_2$?

Question 9
Day 1/Track1



The ray fans above are for a system with no tilt or defocus and are shown for the case of H=1.0 (the edge of the field). The system is operating at a wavelength of 500 nm, with R=20 mm and r_c=5 mm.

(70%)

a) How much defocus would you introduce into the system to give the "best" image. Explain your answer including a description of your criteria for best image. Be sure to give the sign and value of w_{020}. Sketch what the ray fans will look like once you include your defocus.

(30%)

b) The solution found in part (a) is for H=1. What happens to the image quality on axis (H=0) when you introduce this defocus? Explain.

Question 10
Day 1/Track 1

A radiometer that operates over the 1.0 to 15.0 μm spectral range views a set of melting point blackbodies all with emissivities in excess of 0.9999. The results of these measurements are shown in the table at right. The radiometer's field of view is much smaller than the size of the blackbody exit ports and the radiometer and blackbodies are operated in a thermal vacuum chamber.

Blackbody Temperature (K)	Sensor output (millivolts)
576.965	140.20
686.130	260.40
815.951	740.80
970.335	2661.60

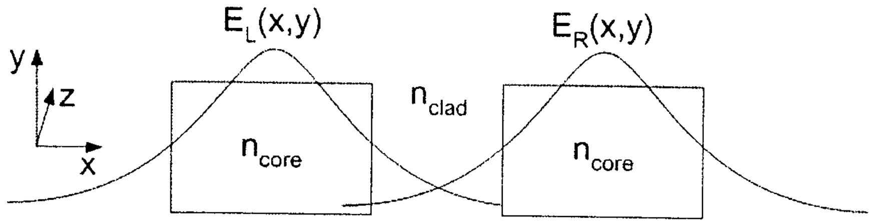
(40%)
a) The output of the radiometer viewing an unknown sample reports 1541.20 millivolts. What temperature is the sample?

(20%)
b) A set of thermometers placed in and on the sample indicate the temperature is 970.335 K for the measurement above of 1541.20 millivolts. What is the emissivity of the sample?

(30%)
c) Describe a set of measurements with this radiometer that could be made to infer the emissivity without knowing the temperature of the sample. State any extra information that you would want to know and/or assumptions you would make along with your measurements.

(10%)
d) All of the measurements are now made in ambient air conditions at a temperature of 300 K. How would this affect your derivation of the emissivity in part b?

Question 11
Day 1/Track 1
Sp '05



Two parallel, identical, rectangular optical waveguides have coupled optical modes with amplitudes A(z) and B(z) that satisfy:

$$\frac{\partial A}{\partial z} = -j\kappa B \text{ and } \frac{\partial B}{\partial z} = -j\kappa A$$

where κ is a coupling constant.

Optical power is launched into the left hand guide (and only the left hand guide) at z = 0 so that we have $A(0) = A_0$ and $B(0) = 0$.

a. Show that optical power oscillates between waveguides, along z, by finding an explicit expression for A(z). (5 points)

b. The optical power in the left hand guide, at its maximum at z = 0, decreases until the power has completely transferred to the right hand guide at z = L. Find an expression for L. (3 points)

c. The coupling constant is given by

$$\kappa = \int_{\substack{\text{core}\\\text{of right}\\\text{guide}}} \frac{\omega\varepsilon_0}{4}\left(n_{core}^2 - n_{clad}^2\right)\left(E_L^* E_R\right)dxdy$$

where the electric fields $E_L(x,y)$ and $E_R(x,y)$ are the normalized optical modes of the left and right optical waveguides respectively.

Using this expression, describe what will happen to the power transfer length L if the distance between the waveguides is increased. (2 points)

Question 12
Day 1/Track 1

Sp '05

Suppose that a 2-level atom is described by the following time-dependent superposition state when in the presence of a laser field:

$$|\Psi(t)\rangle = c_g(t)|\psi_g\rangle e^{-iE_g t/\hbar} + c_e(t)|\psi_e\rangle e^{-iE_e t/\hbar}.$$

Here, $|\psi_g\rangle$ and $|\psi_e\rangle$ are the atom's ground and excited energy eigenstates (respectively) when the field is not present. For this problem, make use of the terms and parameters given above and in the following list:

Ω_0 (the on-resonance Rabi frequency),
Δ (the detuning of the field from atomic resonance),
Γ (the natural lifetime of the excited state).

If you use any other terms in this problem, be sure to define them!

(a) Evaluate $\langle\psi_g|\psi_e\rangle$. *[1 point]*

(b) Make a plot of $|c_g(t)|^2$ for the range of times $0 \leq t \leq t_1$, where $t_1 = 6\pi/\Omega_0$, $c_g(0) = 0$, and $\Delta = 0$. Label your plot axes, and on each axis, label the range of values associated with this plot (ie, maximum and minimum plotted values for $|c_g(t)|^2$ and t). *[2 points]*

(c) Make a new plot of $|c_g(t)|^2$ for times $0 \leq t \leq t_1$, where again $t_1 = 6\pi/\Omega_0$ and $c_g(0) = 0$, but now $\Delta = \sqrt{3}\Omega_0$. What is the generalized Rabi frequency for this case? Again, label the range of values for the plotted points. *[2 points]*

(d) If $c_g(t_2) = -1/2$ for some time t_2, what is the probability of finding the atom in the *excited* state at time $t = t_2$? Provide a number for your answer. *[1 point]*

(e) For the case with $c_g(t_2) = -1/2$, write an expression for $\langle E\rangle$, the energy expectation value for the state $|\Psi(t_2)\rangle$ at time $t = t_2$. Write your answer in terms of E_g and E_e only. *[1 point]*

(f) If the laser turns off at time t_3, and $c_e(t_3) = 1$, write an expression for $|c_e(t > t_3)|^2$ that accounts for spontaneous emission. (Note that this is c_e, *not* c_g.) Check that your answer makes sense in the limits $t = t_3$ and $t = \infty$. *[1 point]*

(g) If a laser beam propagates through a gas of these two-level atoms, what is the maximum possible steady-state value of the ratio N_e/N_{tot}, where N_e is the steady-state population density of the excited state, and N_{tot} is the total population density? *[1 point]*

(h) Can a gas of these two-level atoms provide steady-state gain and amplify a laser beam propagating through it? *[1 point]*

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$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

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$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

$\nabla \times \nabla\phi = 0$

$\oint_S (\mathbf{F} \cdot \mathbf{n})\, da = \int_V (\nabla \cdot \mathbf{F})\, d^3x$

$\oint_C \mathbf{F} \cdot d\ell = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da$

$\oint_S \phi\mathbf{n}\, da = \int_V \nabla\phi\, d^3x$

$\oint_S \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_V [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

$\oint_S (\mathbf{n} \times \mathbf{F})\, da = \int_V (\nabla \times \mathbf{F})\, d^3x$

This is a problem about Markov chains and a two-level atom interacting with an electric field at frequency ω.The state of the atom at time t is given by

$$\Psi(\mathrm{r},t) = a_1(t)\,\Phi_1(\mathrm{r}) + a_2(t)\,\Phi_2(\mathrm{r})$$

The functions Φ_1 and Φ_2 are states of the Hamiltonian of the atom.

If the system is in state 1 at $t=0$, then $a_1(0)=1$ and $a_2(0)=0$ and

$$\begin{aligned} a_1(t) &= \left[\cos\left(\frac{\Omega t}{2}\right) + \frac{i\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right]\exp\left(\frac{-i\Delta t}{2}\right) = a_{11}(t) \\ a_2(t) &= \frac{i\chi}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\exp\left[\frac{-i(\Delta+2\omega)t}{2}\right] = a_{12}(t) \end{aligned}$$

If the system is in state 2 at $t=0$, then $a_1(0)=0$ and $a_2(0)=1$ and

$$\begin{aligned} a_1(t) &= \frac{i\chi}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\exp\left(\frac{-i\Delta t}{2}\right) = a_{21}(t) \\ a_2(t) &= \left[\cos\left(\frac{\Omega t}{2}\right) - \frac{i\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right]\exp\left[\frac{-i(\Delta+2\omega)t}{2}\right] = a_{22}(t) \end{aligned}$$

The constants χ and Δ depend on the atom. The constant Ω is given by

$$\Omega = \sqrt{\chi^2+\Delta^2}$$

We want to study what happens when we repeatedly observe the state of the system. For Ψ as given above, the probabilities, at time t, that we observe state 1 or state 2 are given by

$$p_1(t) = |a_1(t)|^2 \text{ and } p_2(t) = |a_2(t)|^2$$

Once state 1 or state 2 has been observed the two-level atom is, at that moment, in that state. If the clock is reset to $t=0$ at that moment, then the equations (1) or (2) above give the time dependence from that moment onward.

a) (20%) Suppose that the atom is initially in state 1 at $t=0$. We then observe the state at times $n\delta t$ for $n=1,2,\ldots$. Let s_n be the state variable, so that $s_n=1$ or $s_n=2$ depending on the state that we observe at $t=n\delta t$. Note that s_n is a random variable for each n. Find the probabilities for the first observation $\Pr(s_1=1)$ and $\Pr(s_1=2)$

b) (20%) We observe after another δt. Find the conditional probabilities for this observation given the first observation, i.e.

$$\begin{aligned} &\Pr(s_2=1 \mid s_1=1) \text{ and } \Pr(s_2=1 \mid s_1=2) \\ &\Pr(s_2=2 \mid s_1=1) \text{ and } \Pr(s_2=2 \mid s_1=2) \end{aligned}$$

c) (20%) Now use a) and b) to find the probabilities for the second observation $\Pr(s_2=1)$ and $\Pr(s_2=2)$. There is no need to simplify the expressions you get.

d) (20%) Show that the stochastic process $s_1, s_2, \ldots$ is a Markov chain and find the transition matrix $\mathbf{P}$

e) (10% point) Define a probability vector for the n^{th} observation by

$$\mathbf{p}^{(n)} = \begin{pmatrix} \Pr(s_n = 1) \\ \Pr(s_n = 2). \end{pmatrix}$$

Write down the matrix equation that relates $\mathbf{p}^{(n+1)}$ to $\mathbf{p}^{(n)}$

f) (10% point) We can define the initial probability vector as

$$\mathbf{p}^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Write the matrix equation that relates $\mathbf{p}^{(n)}$ to $\mathbf{p}^{(0)}$.

Question 2
Day 1/Track II

We are constructing a prefix code that assigns binary numbers to letters. We will number the letters and that will be a random variable X. There are eight letters and they occur with probabilities given by

Letter	X	$P(X)$
a	1	.1
b	2	.15
c	3	.2
d	4	.15
e	5	.1
f	6	.05
g	7	.05
h	8	.2

a) (20%) Find the entropy $E(X)$ for this probability distribution
b) (20%) Find a binary Huffman code for this problem
c) (20%) Verify that the average codeword length $L(C)$ for your code satisfies

$$E(X) \leq L(C) \leq E(X) + 1$$

d) (20%) What is the minimum average codeword length that we can possibly achieve?

e) (20%) Explain in general terms what block encoding is and why we would use it

Question 3
Day 1/Track II

Consider lattice vibrations of the 1-dimensional monatomic lattice. Let the mass of the atoms be M. The harmonic nearest-neighbor potential is modelled by classical springs with spring constant f. Specify Newton's equation for the displacement of atom n and solve the equation using the ansatz

$$u_n(t) = A\, e^{-i(\Omega t - kna)}$$

Derive the dispersion relation $\Omega = \Omega(k)$. Use the identity $\frac{1}{2}(1 - \cos(ka)) = \sin^2(ka/2)$. Express the value of Ω at the zone boundary of the first Brillouin zone in terms of M and f. Sketch the dispersion in the first Brillouin zone, clearly labelling all special points and values.

Question 4
Day 1/Track II

The definition of a lattice in real space is based on the primitive translation vectors $\vec{a}_i$ $(i = 1, 2, 3)$. The definition of the reciprocal lattice is based on the primitive translation vectors in reciprocal space, $\vec{b}_i$ $(i = 1, 2, 3)$, with

$$\vec{b}_1 = \frac{2\pi}{V_c}\,(\vec{a}_2 \times \vec{a}_3)$$

Specify $\vec{b}_2$ and $\vec{b}_3$ and prove that the term "reciprocal lattice" is indeed appropriate, i.e. that the reciprocal of the reciprocal lattice is the original lattice. For the sake of time, restrict your proof to one (out of three) vectors.

Instructions (mandatory):
Starting from the reciprocal primitive translation vectors $\vec{b}_i$, one can form the three reciprocal-reciprocal primitive translation vectors, and denote them by $\vec{c}_i$ $(i = 1, 2, 3)$. Again, for the sake of time, restrict yourself to $\vec{c}_1$. Denote by V_b the volume of the the primitive cell in reciprocal space (which is spanned by the $\vec{b}_i$). Using the general relation

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

show that $\vec{c}_1 = \vec{a}_1$. Indicate $\vec{A}$, $\vec{B}$ and $\vec{C}$ whenever you use this rule. Do not evaluate the cross products explicitly in terms of Cartesian coordinates. As part of your proof, you will need to show that $V_b = (2\pi)^3/V_c$.

(There are 5 parts, each is worth 20 percent.)
The equation of the motion for the dimensionless intensity I of a single-mode, homogeneously broadened laser is

$$\frac{dI}{dt} = 2I\left[gS(I,\nu) - \kappa\right], \tag{1}$$

where

$$S(I,\nu) = \frac{1}{1 + I\mathcal{L}(\omega - \nu)}$$

and $\mathcal{L}(x)$ is the Lorentzian

$$\mathcal{L}(x) = \frac{\gamma^2}{\gamma^2 + x^2}.$$

I.

What are the physical meanings of g, $S(I,\nu)$, γ and κ. Explain the dependence of $S(I,\nu)$ on the intensity I and the detuning between the laser frequency ν and the atomic transition frequency ω.

II.

Solve Eq. (1)) in steady state and discuss your result in physical terms.

III.

At resonance $\omega = \nu$, carry out a linear stability analysis to determine when each solution is stable/unstable, and when.

IV.

Explain in precise physical terms the meaning of the statement "saturated gain = loss" and the context in which it applies.

V.

Make an accurate sketch that clearly illustrates the various regimes of growth of the laser intensity as a function of time.

Question 6
Day 1/Track II

Prelim question related to Optical Sciences 546
Spring, 2005

The Fourier transform $P(\xi)$ of the PSF $p(x)$ of a 1D linear shift-invariant system is often called the transfer function because it describes how a single complex exponential of the form $\exp(2\pi i\xi x)$ is transferred through the system. In this problem we study a closely related quantity called the *modulation transfer function* or MTF. It is defined by

$$MTF(\xi) = \frac{|P(\xi)|}{P(0)}.$$

To understand the reason for the name of this function, we consider as input to the system a real cosinusoidal pattern described by

$$f(x) = A + B\cos(2\pi\xi_0 x),$$

where $A \geq B$ so that $f(x) \geq 0$ for all x. With this choice of constants, $f(x)$ can represent a brightness or radiance pattern of some object (in 1D, of course). The *modulation* of a periodic pattern such as this can be defined by

$$M_f \equiv \frac{f_{max} - f_{min}}{f_{max} + f_{min}},$$

where f_{max} and f_{min} are, respectively, the maximum and minimum values of $f(x)$.

(a) Compute M_f in terms of A and B.

(b) Decompose $f(x)$ into a sum of complex exponentials and determine the effect of the system on each term. (Recall that $\exp(2\pi i\xi x)$ is an eigenfunction of the system and $P(\xi)$ is the corresponding eigenvalue.)

(c) Add up the terms on the output of the system to get an expression for the image $g(x)$. You may want to use the fact that $P(-\xi) = P^*(\xi)$ since $p(x)$ is assumed to be real, and it may be useful to write the transfer function in modulus-phase form as

$$P(\xi) = |P(\xi)| \exp[i\Phi(\xi)].$$

(d) Compute the modulation of $g(x)$, denoted M_g and defined analogously to M_f.

(e) Finally, show that $M_g/M_f = MTF(\xi_0)$. Thus the MTF is the factor by which the modulation is reduced in going through the system.

Note: All parts carry equal weight

TRACK II

WRITTEN PRELIM EXAM – SECOND DAY

Spring 2005

February 15, 2005
8:30 a.m. to 12:00 p.m.

Answer all six questions. Start each answer on a new page.

In the upper right hand corner of each sheet you hand in, put your code number (NO NAMES PLEASE) and the problem number. Staple together all sheets for a given problem.

Insert your answers into the manila envelope supplied and note which problems you have answered on the outside of the envelope.

The following are some helpful items:

$h = 6.625 \times 10^{-34}\ \mathrm{J \cdot s} = 4.134 \times 10^{-15}\ \mathrm{eV \cdot s}$

$e = 1.6 \times 10^{-19}\ \mathrm{C}$

$c = 3.0 \times 10^{8}\ \mathrm{m/s}$

$k_B = 1.38 \times 10^{-23}\ \mathrm{J/K}$

$\sigma = 5.67 \times 10^{-8}\ \mathrm{W/K^4 \cdot m^2}$

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}$

$\mu_0 = 1.26 \times 10^{-6}\ \mathrm{H/m}$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = 2 \cos^2 A - 1$

$\cos 2A = 1 - 2 \sin^2 A$

$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$

$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$\nabla\phi\psi = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\phi$

$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

$\nabla \cdot (\nabla \times \mathbf{F}) = 0$

$\nabla \times (\phi\mathbf{F}) = \phi(\nabla \times \mathbf{F}) + \nabla\phi \times \mathbf{F}$

$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

$\nabla \times \nabla\phi = 0$

$\oint_{\mathrm{S}} (\mathbf{F} \cdot \mathbf{n})\, da = \int_{\mathrm{V}} (\nabla \cdot \mathbf{F})\, d^3x$

$\oint_{\mathrm{C}} \mathbf{F} \cdot d\ell = \int_{\mathrm{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da$

$\oint_{\mathrm{S}} \phi\mathbf{n}\, da = \int_{\mathrm{V}} \nabla\phi\, d^3x$

$\oint_{\mathrm{S}} \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_{\mathrm{V}} [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

$\oint_{\mathrm{S}} (\mathbf{n} \times \mathbf{F})\, da = \int_{\mathrm{V}} (\nabla \times \mathbf{F})\, d^3x$

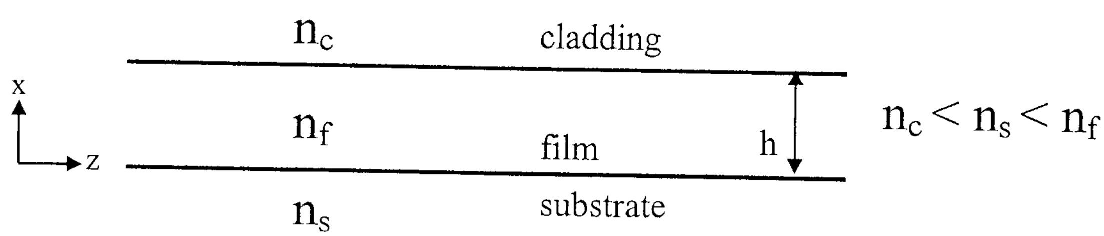
a) Draw a graph illustrating the total dispersion of a typical single mode optical fiber within the wavelength range of 1100 – 1700 nm. Your graph should show approximate values of the dispersion. Draw also lines that indicate how the total dispersion is composed of two separate dispersion mechanisms.

(7 points)

b) Mention three different techniques to compensate for the dispersion in single mode fibers (1/2 point each) and briefly explain their operation principle (1/2 point each).

(3 points)

Question 8
Day 2/Track II



Consider an asymmetric 3-layer slab waveguide shown above (y-invariant). Derive the eigenvalue equation for TM (Transverse Magnetic) modes. In your derivation, use notations:

$$\kappa_f = \sqrt{n_f^2 k_0^2 - \beta^2}\,,\ \gamma_c = \sqrt{\beta^2 - n_c^2 k_0^2}\ \text{ and }\ \gamma_s = \sqrt{\beta^2 - n_s^2 k_0^2}\,,$$

where β is the propagation constant, κ_f is the transverse wavevector, and γ_s and γ_c are the decay (attenuation) coefficients in substrate and cladding, respectively. Note: it is not necessary to simplify the equation to its final form.

Question 9
Day 2/Track II

A spinless particle is trapped in a 2D square potential well, $V(x,y) = \begin{cases} 0 & for \quad 0 \le x,y \le a \\ \infty & elsewhere \end{cases}$.

(1) Indicate the four lowest energies, wavefunctions and their degeneracies for this potential. Reminder: for a 1D square well of width a these energies and wavefunctions are $E_n = (n\pi\hbar)^2/2ma^2$ and $\varphi_n(q) = \sqrt{2/a}\sin(n\pi q/a)$.

The potential is now modified by the addition of a "dimple" at the center of the well. The form of this dimple is $W(x,y) = \beta\delta(x-a/2)\delta(y-a/2)$.

(2) Find an approximate expression for the energy of the new ground state. Indicate the regime of validity for your result. The new ground state can be expressed as a superposition of the original stationary states; indicate which of these are NOT present in the superposition and explain why.

(3) Find an approximation for the energy and wavefunctions for the new states corresponding to the original 4th energy level. Again, indicate the regime of validity for your result.

(4) Assume the particle is in the ground state of $V(x,y)$ at $t=0$, at which time the perturbation $W(x,y)$ is suddenly applied. Assuming $\beta << (\pi\hbar)^2/8m$, sketch the approximate time dependence of the population in the states corresponding to the 2nd through 4th energy level(s). Indicate general behavior only; you do not need to do any explicit calculations, or show the precise scale on the time and population axes. In each case explain how the particular time dependence arises.

Question 10
Day 2/Track II

A physical system consists of two spin-1/2 particles prepared in the state

$$|\Psi\rangle = \alpha|+\rangle_A|+\rangle_B + \beta|+\rangle_A|-\rangle_B + \gamma|-\rangle_A|+\rangle_B + \delta|-\rangle_A|-\rangle_B,$$

where $|\pm\rangle$ are the eigenstates of σ_z with eigenvalues ± 1

(1) Assume that the values of the probability amplitudes α, β, γ and δ are completely unknown – they can be *anything* consistent with $\langle\Psi|\Psi\rangle = 1$. Find the density matrix for the two-spin system in the basis $|x\rangle_A|y\rangle_B$, $x, y = \pm 1$. Does it correspond to a pure state or a statistical mixture? Explain!

(2) We are told that the spins are associated with two indistinguishable particles. What are the possible values of α, β, γ and δ? Find the density matrix for the two-spin system in this case. Does it correspond to a pure state or a statistical mixture? Explain!

(3) We now measure σ_x for one of the spins and get the result $+1$. Find the state vector for the remaining spin. What are the probabilities of getting the results ± 1 when, for this remaining spin, we measure the observables σ_x and σ_z respectively.

(4) Rather than measuring we throw one of the spins away. Find the density matrix for the remaining spin. Does it correspond to a pure state or a statistical mixture? What does this tell us about the nature of the two-particle state.

The formulae for the reflection coefficients for s-polarized and p-polarized incident plane-wave fields at a planar dielectric interface of two media are

$$r_s(\theta) = \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}}, \quad r_p(\theta) = \frac{-n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}},$$

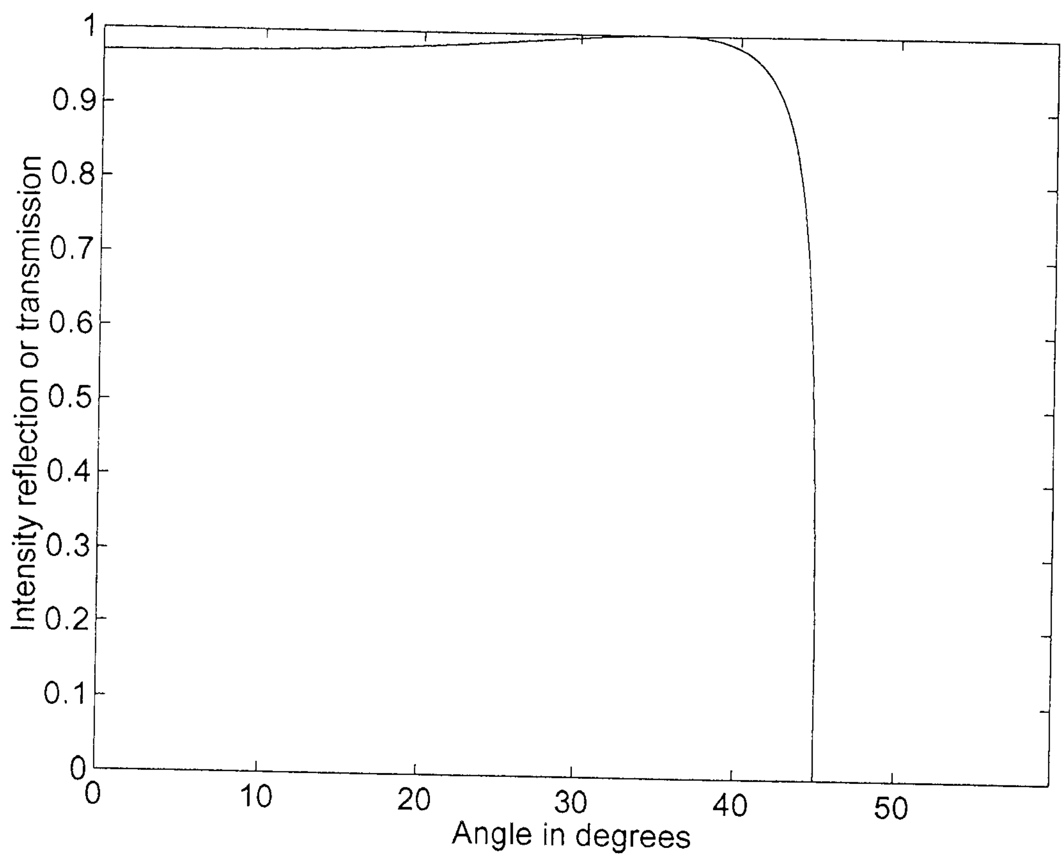
where $n = n_2/n_1$ is the relative refractive index, n_1 being the refractive index of the medium the field is incident from and n_2 the refractive index of the second medium, and θ measures the angle of incidence with respect to the normal to the plane of the interface in the incident medium.

(a) Explain what is meant by the term plane of incidence, and how s-polarized and p-polarized incident fields are defined with respect to the plane of incidence. (2 points)

(b) The wave vector of the plane-wave field of frequency ω incident on the interface is denoted $\vec{k}$, and the reflected and transmitted wave vectors as $\vec{k}'$ and $\vec{k}''$, respectively. By equating the components of each wave vector resolved along the interface, prove that the reflected angle equals the incident angle $\theta' = \theta$, and Snell's law of refraction $n_1 \sin\theta = n_2 \sin\phi$, with ϕ the angle of the transmitted wave vector with respect to the normal to the plane of the interface in the second medium. (3 points)

(c) Using either the expression for $r_s(\theta)$ or $r_p(\theta)$ derive an expression for the critical angle for total internal reflection θ_{cr}, and explain any conditions on the relative refractive-index n to observe total internal reflection. (2 points)

(d) The figure on the next page shows (i) either the intensity reflection R or intensity transmission T, for (ii) either internal of external reflection at a dielectric interface as a function of incident angle θ, and for (iii) either an s-polarized or p-polarized incident field. By inspecting the curve explain to which combination of the above options (i),(ii) & (iii) the plotted curve corresponds. (3 points)


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Intensity reflection or transmission
0
10
20
30
40
50
Angle in degrees

Question 12
Day 2/Track II

For this question consider a plane-wave monochromatic electric field propagating along the z-axis through free space

$$\vec{E}(\vec{r},t) = \vec{e}_x A_x \cos(kz - \omega t + \phi_x) + \vec{e}_y A_y \cos(kz - \omega t + \phi_y), \qquad (1)$$

where $\vec{e}_x$ and $\vec{e}_y$ are the unit vectors in the x and y directions.

(a) Obtain the plane-wave amplitude $\vec{E}(\vec{k},\omega)$ for the above electric field. (1 point)

(b) Using your solution from part (a) construct the normalized Jones vector $\bar{E}$ for the electric field. (1 point)

(c) Consider an optical element with Jones matrix M that leaves an x-polarized field unchanged but imparts a phase-shift ψ and a real amplitude change β to a y-polarized field. Based on this information write down a suitable Jones matrix M for the optical element. (2 points)

(d) Use your solution from part (b) as the input to the optical element with Jones matrix M. Then using the Jones calculus determine how to choose the parameters ψ and β of the optical element so that it would convert the input field into a right handed circular polarized output. (2 points)

(e) Next assume that the optical element from part (c) is preceded by a quarter-wave plate. Using the Jones calculus calculate the polarization state at the output using the Jones vector from part (b) as the input and the values of ψ and β from part (d). (2 points)

(f) Explain whether or not the ordering of the quarter-wave plate and the optical element from part (c) is relevant. (2 points)