WRITTEN PRELIM EXAM – FIRST DAY

Fall 2007

September 25, 2007
8:30 a.m. to 12:00 p.m.

Answer all six questions. Start each answer on a new page.

In the upper right hand corner of each sheet you hand in, put your code number (NO NAMES PLEASE) and the problem number. Staple together all sheets for a given problem.

Insert your answers into the manila envelope supplied and note which problems you have answered on the outside of the envelope.

The following are some helpful items:

$h = 6.625 \times 10^{-34}\ \mathrm{J\cdot s} = 4.134 \times 10^{-15}\ \mathrm{eV\cdot s}$

$e = 1.6 \times 10^{-19}\ \mathrm{C}$

$c = 3.0 \times 10^{8}\ \mathrm{m/s}$

$k_B = 1.38 \times 10^{-23}\ \mathrm{J/K}$

$\sigma = 5.67 \times 10^{-8}\ \mathrm{W/K^4 \cdot m^2}$

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}$

$\mu_0 = 1.26 \times 10^{-6}\ \mathrm{H/m}$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$2\cos A \cos B = \cos(A-B) + \cos(A+B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$\sin 2A = 2\sin A \cos A$

$\cos 2A = 2\cos^2 A - 1$

$\cos 2A = 1 - 2\sin^2 A$

$\sin^2(\frac{A}{2}) = \frac{1}{2}(1 - \cos A)$

$\cos^2(\frac{A}{2}) = \frac{1}{2}(1 + \cos A)$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$\nabla\phi\psi = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F})$

$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla\cdot\mathbf{F}) + \mathbf{F}\cdot\nabla\phi$

$\nabla \cdot (\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\nabla\times\mathbf{F} - \mathbf{F}\cdot\nabla\times\mathbf{G}$

$\nabla \cdot (\nabla \times \mathbf{F}) = 0$

$\nabla \times (\phi\mathbf{F}) = \phi(\nabla\times\mathbf{F}) + \nabla\phi\times\mathbf{F}$

$\nabla \times (\mathbf{F}\times\mathbf{G}) = \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$

$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$

$\nabla \times \nabla\phi = 0$

$\oint_{\mathrm{S}}(\mathbf{F}\cdot\mathbf{n})\, da = \int_{\mathrm{V}}(\nabla\cdot\mathbf{F})\, d^3x$

$\oint_{\mathrm{C}}\mathbf{F}\cdot d\ell = \int_{\mathrm{S}}(\nabla\times\mathbf{F})\cdot\mathbf{n}\, da$

$\oint_{\mathrm{S}}\phi\mathbf{n}\, da = \int_{\mathrm{V}}\nabla\phi\, d^3x$

$\oint_{\mathrm{S}}\mathbf{F}(\mathbf{G}\cdot\mathbf{n})\, da = \int_{\mathrm{V}}[\mathbf{F}(\nabla\cdot\mathbf{G}) + (\mathbf{G}\cdot\nabla)\mathbf{F}]\, d^3x$

$\oint_{\mathrm{S}}(\mathbf{n}\times\mathbf{F})\, da = \int_{\mathrm{V}}(\nabla\times\mathbf{F})\, d^3x$

Day 1, Question 1

15% a) Describe in words the meaning of Maxwell's first equation, $\nabla \cdot \boldsymbol{D} = \rho$. What is ρ, what is $\nabla \cdot$, what is $\boldsymbol{D}$, and how is $\boldsymbol{D}$ related to the electric field $\boldsymbol{E}$?

20% b) Describe in words the meaning of Maxwell's second equation, $\nabla \times \boldsymbol{H} = \boldsymbol{J} + \partial \boldsymbol{D}/\partial t$. What is $\boldsymbol{H}$, what is $\nabla \times$, what is $\boldsymbol{J}$, and how is $\boldsymbol{J}$ related to charge density ρ and the velocity V of these charges?

20% c) Starting with Maxwell's first and second equations, derive the "charge continuity" equation.

15% d) Describe in words the meaning of Maxwell's third equation, $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$. What is $\boldsymbol{B}$, and how is it related to $\boldsymbol{H}$? Under what circumstances can one define a scalar potential ψ such that the electric field may be expressed as $\boldsymbol{E} = -\nabla \psi$?

15% e) Describe in words the meaning of Maxwell's fourth equation, $\nabla \cdot \boldsymbol{B} = 0$.

15% f) Are Maxwell's third and fourth equations consistent with each other in the same sense that the first and second equations were shown to be consistent in part (c) above? Is the fourth equation implicit in the third equation?

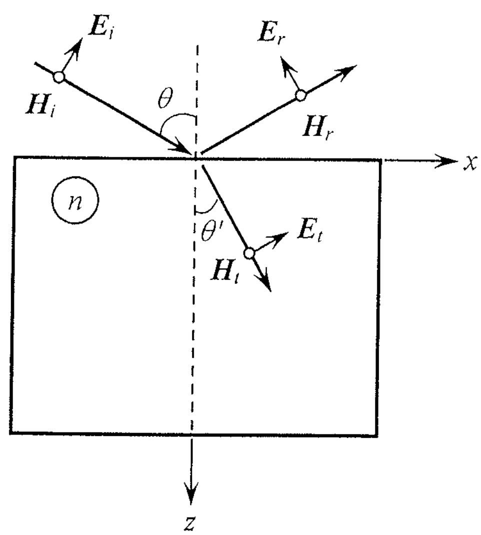
Hint: For an arbitrary vector field $A(r, t)$, application of Stokes' theorem to a closed surface yields $\nabla \cdot (\nabla \times A) = 0$.

A monochromatic plane-wave (wavelength = λ) is incident at an angle θ on a semi-infinite dielectric medium of refractive index n, as shown in the figure. The medium of incidence is free-space, the refractive index n is real-valued, the plane of incidence is xz, and the incident beam's state of polarization is linear, with the E-field confined to the plane of incidence (i.e., p-polarized). In the figure, the various E and H-fields are identified with subscripts i (incident), r (reflected), and t (transmitted). The transmitted beam's angle with the surface normal is θ'.

40% a) Write the complex waveforms of the E- and H-fields for the incident, reflected, and transmitted beams. (Use θ, θ', n, the speed of light $c = 1/\sqrt{\mu_0 \varepsilon_0}$, and the impedance of the free-space $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ to minimize the number of independent variables at this stage. Z_0 is the ratio E/H for a plane-wave in the free space.)

40% b) Use the relevant boundary conditions at the entrance surface to determine the relation between θ and θ' (i.e., Snell's law), and also to relate the various components of the E- and H-fields to each other.

20% c) Solving the set of equations obtained in (b), determine the Fresnel reflection and transmission coefficients at the dielectric surface.



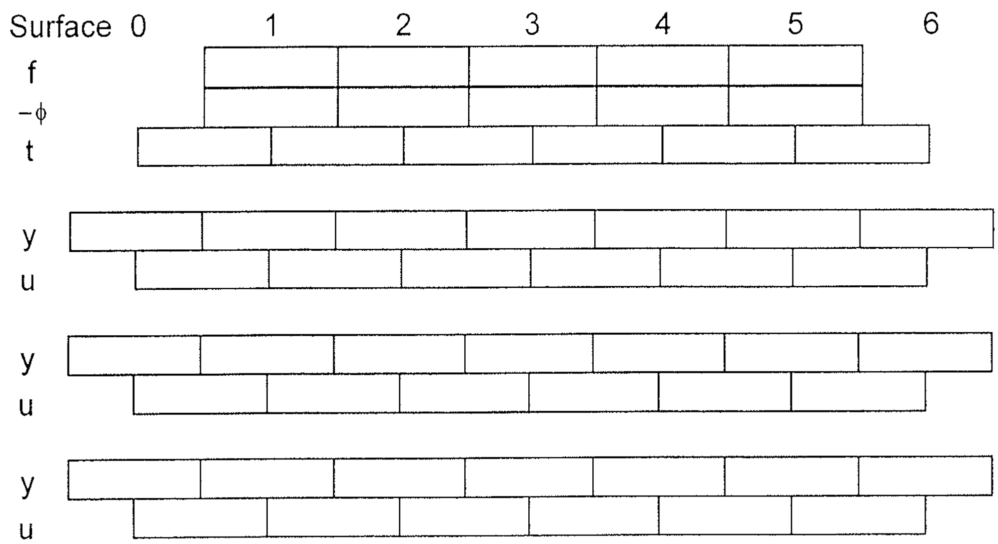
Day 1, Question 3

An f/5.6 lens system is constructed with two thin lenses in air.

The two thin lenses are separated by 80 mm.
The system has a focal length of 200 mm
The system provides a back focal distance or working distance of 70 mm.
The system stop is located halfway between the two thin lenses.
The object is at infinity.
The lens is unvignetted when used with a detector that has a diameter of 40 mm.

Determine the focal lengths and diameters of the two thin lenses, the stop diameter, and the element spacings.

A thin lens raytrace sheet is attached for your convenience.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Day 1, Question 4

A) Write down the formula for the Lagrange invariant Ж and define all quantities involved. 20%

B) Show that the Lagrange invariant is invariant upon transfer. 40%

C) Show that the Lagrange invariant is invariant upon refraction. 40%

Day 1, Question 5

In this problem, you will work with a two-level atom interacting with a single mode of an optical cavity. The cavity axis is along the x direction, and the atom is assumed to be moving in the $-\mathbf{y}$ direction. The atom interacts with the field in the time interval between when it enters the optical cavity (on the $+\mathbf{y}$ side) and exits the cavity (on the $-\mathbf{y}$ side). The time-dependent state of the atom can be written as

$$|\Psi_{atom}(t)\rangle = c_g(t)\,|g\rangle \; + \; c_e(t)\,|e\rangle,$$

where $|g\rangle$ represents the atomic ground state and $|e\rangle$ the atomic excited state. The time dependent state of the electromagnetic field of the cavity mode can be written as

$$|\Psi_{field}(t)\rangle = c_0(t)\,|0\rangle \; + \; c_1(t)\,|1\rangle \; + \; c_2(t)|2\rangle \; + \ldots$$

where $|n\rangle$ represents a photon number state with n photons. The superposition coefficients in this problem are assumed to be normalized in the usual way. In the following questions, you may **neglect phase factors** for all probability amplitudes, treating all probability amplitudes as if they are real and positive. Assume that the optical frequency of the mode is equal to the atomic resonance frequency.

Suppose that at time t=0, an experiment is arranged such that the atom is prepared in an initial state $|\Psi_{atom}(t=0)\rangle = |e\rangle$, and the field is in an initial state $|\Psi_{field}\,(t=0)\rangle = |0\rangle$. The atom enters the cavity starting at time t=0, and immediately starts to interact with the field for a total time of t_{int}, before it exits out of the cavity. Assume that the spontaneous emission time of the excited atom is long enough that it can be completely ignored in all questions below.

Assume that the Rabi oscillation frequency for the atom-field interaction is independent of photon number, and is given by the quantum Rabi oscillation frequency $\Omega_Q = 6.28 \times 10^6\ s^{-1}$.

(a) 3 points. Sketch P_{eg} vs t for t>0, where P_{eg} is the probability of finding the atom in its ground state if it starts in its excited state at t = 0. Let your sketch cover 2 full Rabi oscillations. Indicate the time t_1 = 500 ns on your sketch.

(b) 2 points. If the atom interacts with the field for a total time of t_{int} = 500 ns, give the values of c_e, c_g, c_0, and c_1 corresponding to the time that the atom leaves the cavity.

(c) 3 points. Now suppose that the atom moves through the cavity twice as fast as in the above case, such that it interacts with the cavity mode for a time t_{int} = 250 ns before exiting the cavity. Give the values of c_e, c_g, c_0, and c_1 corresponding to the time that the atom leaves the cavity.

(d) 2 points. If the atom is in its **ground** state when it enters the cavity, give the values of c_e, c_g, c_0, and c_1 corresponding to the time that the atom exits the cavity 250 ns later.

Day 1, Question 5

1) What are the two level density matrix equations? Write them down to the best of your memory/knowledge. (1pt)

2) What are the populations of the upper and lower levels in terms of state amplitudes? (1pt)

3) What is the polarization ρ_{12} in term of state amplitudes? (1pt)

4) How does the coherence time γ_{12} ($1/T_2$) depend upon γ_{22} ($1/T_1$) for an isolated atom undergoing no collisions? (1pt)

5) Describe or sketch the linear absorption for the $^2S_{1/2}$ to $^2P_{1/2}$ (D_1 transition) of a single stationary sodium atom, neglecting nuclear spin and magnetic field effects. How large is the (natural) linewidth? What is the wavelength or color of the light? (1pt)

6) Describe or sketch the linear absorption for the same transition as in 5), but this time for a sodium atom vapor in a cell held at 200C? Roughly how large is the Doppler broadened linewidth? (1pt)

7) How does Doppler broadening depend upon temperature? To what temperature must one cool the gas to reduce the Doppler width to 1 MHz? (1pt)

8) Can one obtain lasing by pumping a two-level system incoherently; explain why or why not. (1pt)

9) Same question for a coherently driven two-level atom. (1pt)

10) Sketch the Mollow probe absorption/gain spectrum when the pump laser is detuned from resonance for a stationary atom. (1pt)

(10 pts total)

Note: $m_{Na} = 3.82 \times 10^{-25}$ kg $\qquad k = 1.38\ 10^{-23}$ J/K

Day 1, Question 6

1. Write down the time-dependent Schrodinger equation for the wave function $\Psi(x,t)$ describing the quantum motion of a particle of mass m in a one-dimensional harmonic oscillator (HO) potential of natural frequency ω_0 and which is centered on x=0. (1)
2. Write down the expression for the energy eigenvalues E_n for the quantum HO. (1)
3. Sketch the form of the ground and first excited, energy eigenstates $\psi_n(x)$, n=0,1 for the quantum HO (here we discard any overall phase factors and take the energy eigenstates to be real). (1)
4. Consider an initial wave function of the form $\Psi(x,t=0) = c_0\psi_0(x) + c_1\psi_1(x)$, with c_0 and c_1 real and greater than zero, $c_0^2 + c_1^2 = 1$, and $c_0 >> c_1$. Argue that this initial wave function is like the ground state but with its center displaced slightly to one side with respect to the origin. (1)
5. Given the initial wave function above write down an expression for the wave function $\Psi(x,t)$ for t>0. (1)
6. Using your solution from part (5) describe how the density profile is changed with respect to the initial profile for times $t=\pi/\omega_0$ and $t=2\pi/\omega_0$. (2)
7. Based on what you have learned above describe how the density profile changes with time. (1)

The Hermite-Gaussian solutions that arise as the energy eigenstates of the quantum HO also appear in the description of the transverse modes of optical resonators. The eigen frequencies ν_{mnq} for a cylindrically symmetric stable optical resonator of length L can be written as

$$\nu_{mnq} = \left(\frac{c}{2L}\right)\left[q + (m+n+1)\frac{\cos^{-1}\sqrt{g_1 g_2}}{\pi}\right]$$

8. Demonstrate that, analogous to the energy eigenvalues of the quantum HO, the frequency eigenvalues of a stable optical resonator are evenly spaced as a function of either of the transverse mode numbers for a fixed longitudinal mode. (1)
9. Show that for a given transverse mode adjacent longitudinal modes are spaced by the free spectral range. (1)

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Day 1, Question 6

1) Sketch a Hanbury-Brown and Twiss apparatus for studying photon statistics. (1pt)

2) Explain what is the second order correlation function $g^{(2)}(\tau)$. (1pt)

3) What is the value of $g^{(2)}(0)$ for the light emitted by a single atom or single quantum dot? Sketch $g^{(2)}(\tau)$. (1pt)

4) On a time axis, indicate a typical pattern of photon arrival times at a detector illuminated by a single-atom source. Same question for a detector illuminated by a hot-filament lamp. Which source is bunched and which antibunched? (1pt)

5) Sketch $g^{(2)}(\tau)$ for the light emitted by 1 million randomly moving atoms. (1pt)

6) If you put a million atoms in a cavity and make a laser, describe or sketch $g^{(2)}(0)$ for the laser output as a function of input pump power. (1pt)

7) Do the same for a single-atom laser with a photon storage time much longer than an atom cycle time (Walther-type). (1pt)

8) Can one measure $g^{(2)}(\tau)$ using a single detector? Explain why or why not. (1pt)

9) Describe what is meant by Rabi oscillations. Sketch the evolution of the Bloch vector for zero detuning between the atom and the coherent field. (1pt)

10) Describe self-induced transparency and how it relates to Rabi oscillations. (1pt)

(10 pts total)

WRITTEN PRELIM EXAM – SECOND DAY

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$2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$\sin 2A = 2\sin A \cos A$

$\cos 2A = 2\cos^2 A - 1$

$\cos 2A = 1 - 2\sin^2 A$

$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$

$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

$\nabla\phi\psi = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

$\nabla \cdot (\phi\mathbf{F}) = \phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla\phi$

$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

$\nabla \cdot (\nabla \times \mathbf{F}) = 0$

$\nabla \times (\phi\mathbf{F}) = \phi(\nabla \times \mathbf{F}) + \nabla\phi \times \mathbf{F}$

$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

$\nabla \times \nabla\phi = 0$

$\oint_{\mathrm{S}} (\mathbf{F} \cdot \mathbf{n})\, da = \int_{\mathrm{V}} (\nabla \cdot \mathbf{F})\, d^3x$

$\oint_{\mathrm{C}} \mathbf{F} \cdot d\ell = \int_{\mathrm{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da$

$\oint_{\mathrm{S}} \phi\mathbf{n}\, da = \int_{\mathrm{V}} \nabla\phi\, d^3x$

$\oint_{\mathrm{S}} \mathbf{F}(\mathbf{G} \cdot \mathbf{n})\, da = \int_{\mathrm{V}} [\mathbf{F}(\nabla \cdot \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}]\, d^3x$

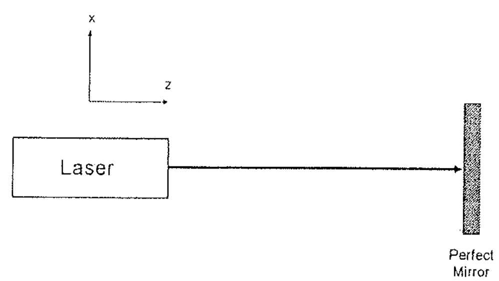
$\oint_{\mathrm{S}} (\mathbf{n} \times \mathbf{F})\, da = \int_{\mathrm{V}} (\nabla \times \mathbf{F})\, d^3x$

Day 2, Question 7

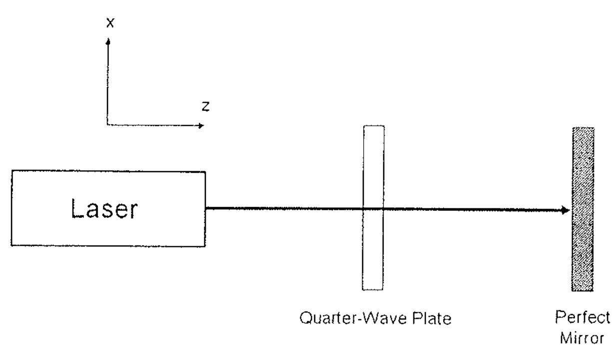
a.) (2.5 pts.) Can white-light fringes be observed for a Michelson interferometer arranged to see Haidinger fringes? (Haidinger fringes are also called fringes of equal inclination.)

b.) (2.5 pts.) Can white-light fringes be observed for a Fizeau interferometer arranged to see Newton's rings?

c.) (2.5 pts.) If fringes are present, sketch the fringe planes in their proper orientation and determine minimum fringe spacing for the following optical geometry. The laser is monochromatic, polarized in the y direction and $\lambda = 500\text{nm}$. The perfect mirror's surface is normal to the collimated laser beam. For the perfect mirror, assume reflection is unity with no phase shifts.

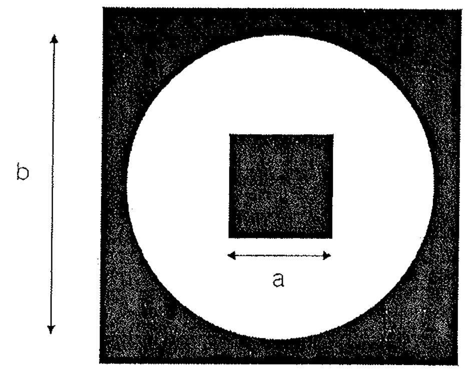


d.) (2.5 pts.) If fringes are present, sketch the fringe planes in their proper orientation and determine minimum fringe spacing for the following optical geometry. The laser is monochromatic, polarized in the y direction and $\lambda = 500\text{nm}$. The perfect mirror's surface is normal to the collimated laser beam. For the perfect mirror, assume reflection is unity with no phase shifts. The quarter-wave plate is aligned to change the linear polarization of the laser into circular polarization.



Day 2, Question 8

An otherwise perfect optical system with a round exit pupil exhibits a square opaque mask in the center of the pupil, as shown below. The diameter of the pupil is b, and each side of the square mask has length a.



a.) (5 pts.) Write a mathematical expression for the absolute value of the electric field amplitude at the focus of the optical system, assuming that the illumination in the pupil is a unit-amplitude on-axis plane wave. Use the symbol f for the focal length and λ for the wavelength.

b.) (5 pts.) Find the ratio of the peak irradiance at the focal plane between the system with the mask and the system without the mask, in terms of a and b and any other parameters that may be necessary.

Day 2, Question 9

Derive the phonon dispersion relation of a monatomic lattice. Specifically, consider only a simple one-dimensional crystal with harmonic nearest neighbor interactions (let f_1 denote the positive and real force constant describing this interaction) as well as next-nearest neighbor interaction (here described by the positive and real force constant f_2).

Proceed as follows.

Specify Newton's equations for the atomic displacements u_j of the j-th atom in the chain. Use a plane-wave-like ansatz (trial function) to describe the eigenmodes of the system (i.e., the modes where all atoms oscillate at the same frequency Ω). Denote the wavevector by k. Requiring non-trivial solutions, solve for the dispersion relation $\Omega(k)$.

In addition to the general dispersion relation $\Omega(k)$, determine the long-wavelength limit of the dispersion and specify the corresponding phase (=group) velocity.

(10 points)

This problem consists of two parts, each with equal weight. Subparts within each each part are equally weighted also.

Part 1: Origin and meaning of effective mass

Newton's law of motion for a particle in free space subject to a force $\mathbf{F}$ can be written as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt},$$

where $\mathbf{p} = m\mathbf{v}$ is the momentum of the particle, m is its mass and $\mathbf{v}$ is its velocity.

An analogous result holds for motion of an electron in an ideal crystal. In that case the electron is a wave, for which one can contruct a wavepacket centered at wavevector $\mathbf{k}$ and moving with group velocity $\mathbf{v}_g$. In that case, it is known that

$$\mathbf{F} = \hbar\frac{d\mathbf{k}}{dt}. \tag{1}$$

(1a) You are not expected to prove this powerful relation, but as a simple example of its use, consider a semiconductor crystal where the conduction band is described by

$$\mathcal{E}(k) = \mathcal{E}_0 + \alpha k^2, \tag{2}$$

where $\mathcal{E}$ denotes energy and k is the magnitude of the wavevector $\mathbf{k}$. Compute the group velocity $\mathbf{v}_g$ for this band model. What is the direction of $\mathbf{v}_g$ relative to $\mathbf{k}$?

(1b) Use elementary calculus to show that

$$dk = \frac{1}{d^2\omega/dk^2} d\left(\frac{d\omega}{dk}\right)$$

and hence

$$\frac{dk}{dt} = \frac{1}{d^2\omega/dk^2}\frac{dv_g}{dt}, \tag{3}$$

(1c) Use (1) and (3) to show that the motion of an electron in a lattice is exactly what one gets in free space except that the mass is replaced by an effective mass m^*:

$$\mathbf{F} = m^*\frac{d\mathbf{v}_g}{dt}. \tag{4}$$

Show that the effective mass is given by

$$\frac{1}{m^*} = \frac{1}{\hbar^2}\frac{d^2\mathcal{E}}{dk^2} \tag{5}$$

and relate m^* explicitly to α.

Part 2: Electron mobility

Part 1 shows that an electron in a perfect crystal will experience a steady acceleration under the influence of a time-independent force. In reality, crystals are not perfect. They may contain impurities and lattice defects that scatter an electron and impede its acceleration, and at finite temperatures there are lattice vibrations (phonons) that also scatter the electrons. We can account for these scattering effects phenomenologically by modifying (4) to

$$\frac{dv_g}{dt} = \frac{1}{m^*}F - \frac{v_g}{\tau_{sc}}\,, \tag{6}$$

where τ_{sc} is the mean time between scattering events.

(2a) Suppose the force is the result of a constant electric field $\mathbf{E}$, so that is $\mathbf{F} = -e\mathbf{E}$, where e is the charge on the electron (a positive number). Show that the steady-state velocity is given by

$$\mathbf{v}_g = -\frac{e\tau_{sc}}{m^*}\mathbf{E} \equiv -\mu\mathbf{E}\,, \tag{7}$$

where μ is called the *mobility* of the electron. In words, mobility is the steady-state drift velocity per unit field.

(2b) Now suppose that the crystal contains n free or nearly free electrons per unit volume, and that they all drift at the velocity given by (7). Compute the current density $\mathbf{J}$ and show that it is a constant times the electric field:

$$\mathbf{J} = \sigma\mathbf{E}\,. \tag{8}$$

The constant σ is called the *conductivity*, and (8) is the microscopic form of Ohm's law.

(2c) State the relationship between conductivity σ and mobility μ. Give the SI units of each.

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Day 2, Question 10

Consider a hypothetical 3-dimensional crystal with one isotropic and parabolic two-fold degenerate energy band ε_k. Specify ε_k, which is characterized by the effective mass m_e (for simplicity, let zero energy be at the bottom of the band). Specify the formula for the distribution function at arbitrary temperature as function of the magnitude of the wavevector ($k = |\vec{k}|$). Also, specify that function for T=0K. In both cases, you don't need to derive the function, just specify its functional form.

Sketch the T=0K distribution as function of k. Determine (by direct integration) the mean energy density as function of the Fermi wavevector. The mean energy density is defined as $\langle \varepsilon \rangle = 2 \int \frac{d^3k}{(2\pi)^3} \varepsilon_k f_k$.

(10 points)

Day 2, Question 10

Parts are equally weighted

(a) Sketch a silicon photodiode and explain how it works.

(b) Sketch and explain the current-voltage characteristic of a silicon photodiode with and without illumination. For the illumination, consider two different wavelengths, $\lambda = 0.5\,\mu$m and $5.0\,\mu$m.

(c) What is meant by the term quantum efficiency for a photodetector? What are its units? Sketch the quantum efficiency of a silicon photodiode as a function of wavelength and explain the physical processes that limit it.

Potentially useful information:

The charge on an electron is 1.6×10^{-19} Coulomb.

The bandgap of silicon at 300 K is 1.12 eV.

The index of refraction of silicon in the visible is approximately 4.

Planck's constant times the speed of light is 1.2398 in units of eV-μm

Boltzmann's constant is 11,600 in units of eV per degree Kelvin

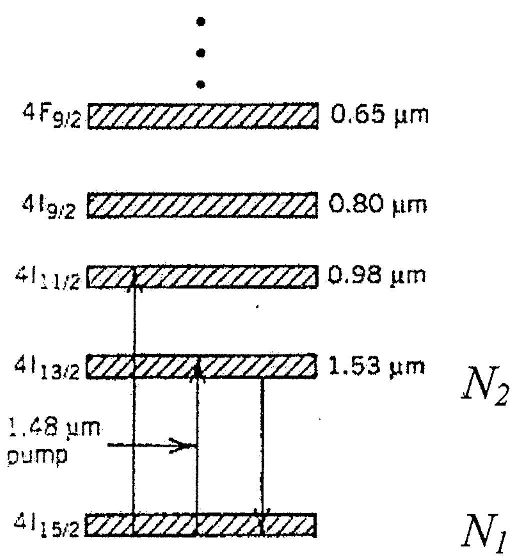
Day 2, Question 11

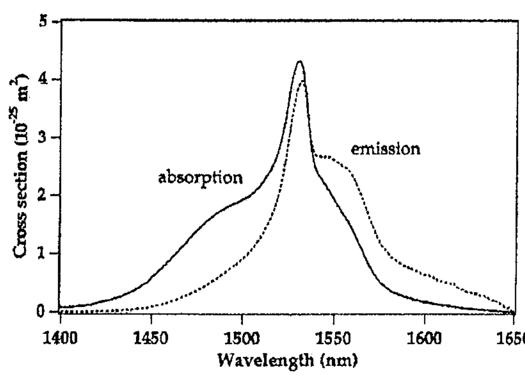
Figures below show the simplified energy-level diagram of Er^{3+} ions in silica host (left) and emission and absorption cross sections for the $^4I_{13/2} \leftrightarrow {}^4I_{15/2}$ transition (right).

a) Consider a simple two level model for an Erbium-Doped Fiber Amplifier (EDFA).

Write the rate equations dN_2/dt and dN_1/dt, in terms of photon fluxes for signal and pump, emission and absorption cross sections, spontaneous emission lifetime of the $^4I_{13/2}$ level and the population densities and of the ground state ($\mathbf{N_1}$) and the excited state ($\mathbf{N_2}$). <u>Write the equations both in the case of 980 nm and 1480 nm pump wavelength.</u>

(4 points)





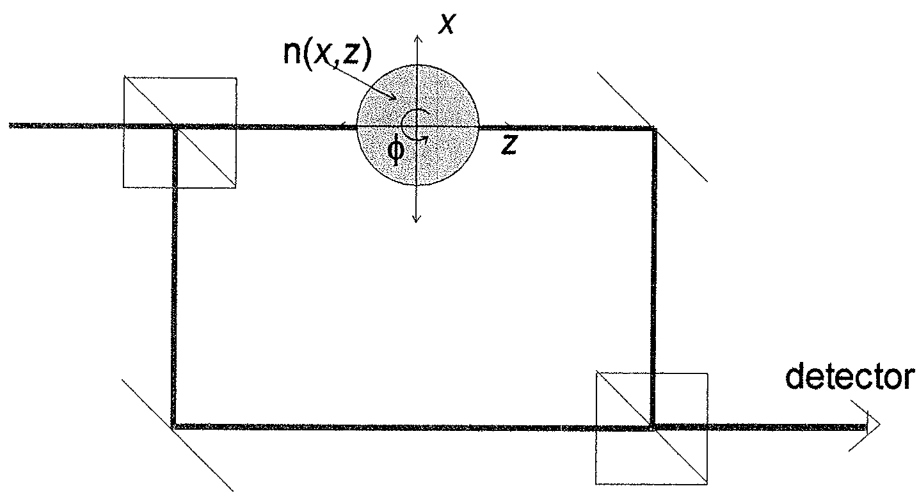
b) Briefly explain why it is possible to use light at 1480 nm wavelength to efficiently pump an EDFA (i.e. to achieve relatively good inversion at longer wavelengths).

(3 points)

c) In terms of noise properties of an EDFA, briefly explain why 980 nm pumping is advantageous compared to 1480 nm pumping.

(3 points)

Day 2, Question 11



The system depicted above is a tomographic system for measuring the distribution of index of refraction in a sample with low index contrast (such as a transparent biological sample). The sample is positioned on a stage that can be translated in x and rotated in ϕ Assume that the maximum extent of the object is much larger than the beam diameter, and ignore the effects of diffraction. You may also assume that the index contrast is low enough that the deviation of the beam due to refraction at the interfaces is negligible. The wavelength of operation is $\lambda = 500\text{nm}$ and the beam is completely temporally coherent. The 50-50 beamsplitters and mirrors are assumed to be ideal.

1. Write an expression for the measured intensity at the detector in this system as a function of the scan variables x and ϕ for an arbitrary object.
2. Assume that the object is described by the index of refraction distribution
$$\left[n(x,z)-1\right]=(n_0-1)\mathrm{cyl}\left(\frac{r}{a}\right),$$
where $r=\sqrt{x^2+z^2}$ and $a = 50\lambda$ is much larger than the beam diameter. Assume that $n_0 = 1.05$. Sketch the output of the detector as the object is translated in the x-direction. Be sure to accurately label the locations of the peaks and nulls.
3. Explain how you would use this system to reconstruct the index of refraction distribution for an unknown sample.
4. Illustrate your method for the sample in part 2. You need not be exact, just illustrate the steps and sketch the appropriate functions at each step.

(Parts are equally weighted)

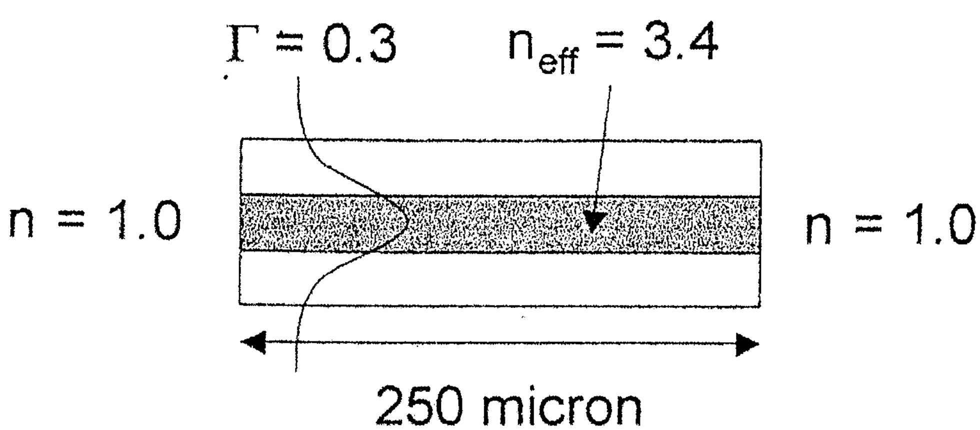
Day 2, Question 12

Consider a Fabry-Perot semiconductor laser sketched below. Estimate the (power) gain coefficient in the active region required for lasing, using the parameters below.

(10 points)

- Active region length: 250 μm.
- The optical confinement factor is 0.3. It gives the fraction of the waveguide mode overlapping with the active region.
- Free carriers in the doped cladding regions absorb light in the cladding regions, which have an absorption coefficient $\alpha_{cl} = 50$ cm^{-1}.

Note that the amplitude reflection coefficient from a cleaved facet is: $r = \dfrac{n_{eff} - 1}{n_{eff} + 1}$.



1. Consider an ideal diffraction-limited imaging system where one is imaging a self-luminous (incoherent) object distribution $f(\underset{\sim}{r}_o)$ through an ideal lens with a circular pupil $p(\underset{\sim}{r}_a) = cyl\left(\frac{r_a}{a}\right)$.
   a. (2 points) What is the continuous image intensity (irradiance) distribution $g(\underset{\sim}{r})$ in the image plane assuming an imaging system magnification of -1, object distance = l image distance = l', and wavelength λ?
   b. (2 points) What is the functional form of transfer function for this incoherent imaging system?
   c. (1 point) What is the maximum (cutoff) spatial frequency for this imaging system if the NA on the image side is 0.25 and the wavelength is 500 nm?
   d. (2 points) If this imaging system is aberrated, with a wavefront error $W(\underset{\sim}{r}_o;\underset{\sim}{r}_a)$ in the pupil, write down a general expression (integral equation) for the image distribution in the image plane.
   e. (2 points) Write down an integral expression for the components g_m of the M x 1 data vector $\underset{\sim}{g}$, given the output distribution $g(\underset{\sim}{r})$, when there is a CCD detector located at the output plane with contiguous square pixels of size d and positions $\underset{\sim}{r}_m = (x_m, y_m)$ (m is the discrete index).
   f. (1 point) If you ignore aberrations, what should the detector spacing d be in order to satisfy the Nyquist sampling condition on the image distribution?